An Introduction
to
Symbolic Logic

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Glossary

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Introduction

The word "logic" comes from the Greek word λογος (*logos*), which is often attached as a suffix to other Greek words, providing us with many modern terms. For example, attaching λογος to ψυχη (*psyche*), meaning soul (as opposed to body), yields "psyche-logos", or "psychology"—the systematic study of mind. Attaching λογος to βιος (*bios*), meaning life, yields "bio-logos", or "biology"—the systematic study of life. And so on for other English words ending in "-logy". The fundamental root, λογος, has a variety of meanings, perhaps the most important of which are word, thought, reason, or explanation.

One might expect that the word *logic* refers to a tried-and-true, or proven, or correct method of reasoning. That *logic* refers to reasoning is easily granted. But that there is one and only one definitive logic—as though a human's reasoning apparatus ought to be structured in exactly one way—is a claim which, to say the least, is highly suspicious. Moreover, when it comes to a formal presentation of logic (or of *a* logic), we are on even shakier ground.

What is the field of logic? We have already noted that logic has something to do with reasoning. Very often the study of logic is divided into sub-fields: ***Deductive logic*** has to do with the relations between premises and conclusions of arguments such that if the premises were true, the conclusions would also have to be true. That is to say, the premises, if true, would give complete and thorough support for the truth of the conclusion. Deduction and deductive techniques will be the main focus of this book. Deduction may be contrasted with ***induction***, wherein the premises of an argument are understood as providing only probable support for the conclusion. (By the way, this use of the term *induction* should not be confused with the proof technique called ***mathematical induction***, which is actually a type of deduction.) Reasoning has also to do with the process of generating hypotheses to be used as explanations or as candidates for conclusions (of either deductions or inductions). This is sometimes called ***abduction***, and there seems to be a special element of creativity in this process, as there is also in ***analogy***, whereby we hope to mark out something in the unknown on the basis of its structural similarity to the already known. In all this—even in deduction—the human imagination plays a prominent role, yet that role has traditionally been minimized or neglected altogether. A more or less rigorous study of the imagination is, these days, pretty much left to the offices of experimental psychology.

The field of logic, insofar as it is a sub-discipline of philosophy, is usually treated in one or both of two ways: (1) In ***informal logic***, everyday arguments expressed in everyday language are analyzed and evaluated. The rules for analysis and evaluation are neither all-encompassing nor very rigorous. A politician's speech, for example, might be carefully studied so that it may be stripped of its hedgings, half-truths, non-promissory promises, non-committal commitments and other baggage so as to lay bare, as much as possible, what is really being said (or not said). We thereby prepare ourselves for life amid the fast-talkers, charlatans and sloppy thinkers who wish to do us out of our time, energy, money or intellectual independence. As a happy bonus, we may be encouraged to be more clear and careful in the arrangements of our own thoughts and in their presentation to other people. (As an unhappy bonus, we may be in a better position to join the ranks of the rogues who prey upon the unsuspecting.)

(2) In ***formal logic***, on the other hand, we seek to formulate strict rules for the expression and manipulation of certain well-defined concepts—to symbolize the concepts and their interrelationships, when possible, so as to be able to generalize and abstract with precision. Formal logics abound. There are incompatible systems, each with its advocates. Since some formal systems take their cues from an analysis of real reasoning, and since real reasoning comes in a variety of flavors and styles, we can expect that various formal systems will behave differently. And indeed they do. Perhaps the most we ought to expect from anything calling itself "logic" is that it be a systematic way of reasoning. But there are systems and there are systems ….

In this book we shall be developing, in a fairly non-rigorous (i.e., easygoing) manner, a system to symbolize (i.e., formalize) some of what can be stated, denied, inferred and otherwise manipulated in English (or in any other natural language). We won't presume to succeed entirely in this task, but we will be able to make some progress.

We will be dealing with a binary logic—a logic which attributes to any statement one of two values, namely, ***TRUE*** or ***FALSE***. We needn't use those two values in particular; any two different values will do. In an electronic circuit, for example, we may speak of a switch being open or closed, or of a voltage level being high or low. We could manufacture many such examples. Still, there are only two values at work here, and so we are dealing with a binary (two-valued) system, and what names we give to those two values is largely a matter of convenience. Digital electronic engineers might (and sometimes do) speak of true and false signals rather than of high and low voltages. And the engineers might choose to equate *high* with *true* and *low* with *false*; but equating *high* with *false* and *low* with *true* will do as well, as long as everything is carried out consistently in order to avoid ambiguities.

Alternatively, we could speak of the two values in terms of only one of them. Thus, instead of *false* we might say something like "not true" or "other than true". Our universe would be conceptually divided into exactly two parts: one part described by means of some convenient term, and the other part simply everything else, whether we chose to give it a name or not.

So much for the binary nature of our logic. We can say something else about it, namely, that it is ***exclusive***. That is, one value excludes the other. If something is true (or high, or made of butter), then it is not false (or low or not made of butter). Exclusivity does not require a binary system: we might have a trinary (three-valued) system such that any value excludes the other two, or a four-valued system such that any value excludes the other three. And so on.

Our system is also ***discrete***. Anything with one of the values has that value completely. (A "partially true" sentence is not allowed in our system.) This is to be contrasted with a logic which allows for partial participation—or ascriptions of partial values, or uncertain ascriptions of values. In a "fuzzy logic", for example, you might say, "On a scale of 0 to 1, Fred's grade of membership in the class of bureaucrats is .7". This is to indicate that the boundary of the class of bureaucrats is not clear—possibly because we're not altogether sure how we want to define "bureaucrat" in all situations, and consequently the concept applies only "fuzzily" to Fred. But in a discrete logic, the sentence "Fred is a bureaucrat" would have exactly one and only one of the allowed values: In a binary logic, the sentence would be either true or false (though we may not know for sure which it is). Or in a three-valued logic, the same sentence might be either true or false or some third value (such as "indeterminate").

Our logic, then—the logic we will study in this book—, will be a discrete, exclusive, binary system. But it is not the only discrete, exclusive, binary system possible.

There are any number of motives for our enterprise: (1) It is fun in its own right. (2) It will encourage precision of thought and expression, thereby shutting out some ambiguities, misunderstandings and fraud. (3) There is the possibility of mechanizing the manipulation of ideas, perhaps following the lead of mathematics, where even the simplest hand-held calculators can do mathematical figuring far faster than humans. Such a mechanization, to the extent that it is successful, might be used in the modeling of some theories of intelligence (although there is still considerable debate about that).

But we must not allow ourselves to be carried away by these rather grand designs. A generous portion of humility is a valuable asset, for we will later on discover enormous difficulties with such an undertaking. The problems with formal—i.e., rigorously symbolic—systems begin to emerge when we try to understand (or to manufacture) their connection with everyday reasoning. It is, for example, not particularly difficult to program a computer to follow a given formal logic system. But it has been, so far, at least, well-nigh impossible to program computers to think (if "think" it may be called) about everyday matters which humans do without conscious attention. Any formal system will end up doing at least a little injustice to the reasoning we perform as a matter of course, because in formal systems we must make everything explicit, whereas ordinary human reasoning leaves much unsaid, and perhaps even unsayable. In addition, we will be able to deal here only with deductive reasoning, leaving nearly untouched the entire areas of induction, analogy and hypothesis which human reasoning seems constantly to make much use of.

As an entertaining beginning to the more rigorous use of deductive thinking, consider the following puzzle.

> The members of an amateur musical sextet are Adams, Brown, Curry, Decker, Espinoza, and Freeman. The instruments they play are bassoon, clarinet, flute, oboe, piano, and saxophone, though not necessarily in that order. Here are some facts about the group:
>
> 1. Adams taught the clarinet to the clarinet player and the bassoon to the bassoon player.
> 2. Curry, Espinoza, and Freeman are self-taught.
> 3. Decker, who is the mother of both the clarinet player and the saxophone player, is in love with the oboe player.
> 4. The oboe player is in love with Adams.
> 5. Espinoza sits between the flute player and Freeman.
> 6. Espinoza's mother is in the audience.
> 7. The piano player is a student of a famous pianist.
>
> Who plays which instrument?

How should we go about finding the answers? Some kind of *systematic* approach would be helpful. In this case, we can systematize the problem by creating a tableau with persons' names identifying the rows, and instruments as columns. (Or you could have names as columns, and instruments as rows.) Then you could employ two symbols, such as "X" and "O": each intersection (cell) can be marked with "X" when it is guaranteed to be impossible, or with "O" when it is known to be necessary. For example, item 5 above implies that neither Espinoza nor Freeman is the flute player. That double inference can be added to the chart. The result so far would be:

	bassoon	clarinet	flute	oboe	piano	saxophone
Adams						
Brown						
Curry						
Decker						
Espinoza			X			
Freeman			X			

You may find it an interesting exercise in deduction to make additional inferences in order to complete the table, thereby providing the solution to the puzzle.

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Sentences and Connectives

1.1 Sentences

We will confine our attention to what we shall call ***sentences***—those propositions, claims, statements, pronouncements or utterances which are either true or false. If something appears to be a sentence, but cannot be true or false, then it is not really a sentence for our purposes. This concept of *sentence* is slightly different from the grammatical concept. Usually we take the following to be proper examples of sentences: “Where are my glasses?” “You bumbling idiot!” “Close the door.” Accordingly, some logicians prefer to use some new term and leave the word *sentence* to the grammarians. Moreover, consider the English sentence “It is raining”. It would not be unreasonable to claim that the German sentence “Es regnet” and the French sentence “Il pleut” express the very same idea as the English sentence, even though different words are being used. So we come quite easily to a distinction between a sentence, which is a particular string of letters (or, if spoken, a particular string of sounds), and, on the other hand, the *meaning* of a sentence. A meaning, which might be the same for different sentences, is sometimes referred to as a *proposition*. Some writers, therefore, prefer to discuss logic in terms of propositions instead of in terms of sentences. For our purposes, though, we may be content to notice this subtlety and then to ignore it, pushing ahead at full pace, using the words “sentence” and “proposition” (and, indeed, “statement” and “claim” as well) interchangeably.

There are some crucial properties which sentences (in our sense of the word) must have, and much of what follows will depend upon these properties.

> ***Law of Identity:*** *Anything is identical with itself. With specific application to sentences, this will be: If a sentence is true, then it is true; and if it is false, then it is false. (What could be simpler?)*
>
> ***Law of Non-Contradiction:*** *If a sentence is true, then it is not false; and if a sentence is false, then it is not true.*
>
> ***Law of Excluded Middle:*** *A sentence is either true or false, but not neither. That is, the only values which sentences may have are TRUE and FALSE; there is no other, or middle, road. A sentence may not be "partly true", or "sort of true" or "meaningless".*

Why should we accept these rules? That is a difficult question. Some authors (although usually not contemporary logicians) insist that those three rules represent the three "laws of thought", without which reasoning could not be carried on. Others find them simply intuitive. Still others find them somewhat suspicious. The Law of Non-Contradiction does seem to be somehow fundamental, since we are primarily interested in dealing with systems of sentences which do not conflict with one another. We ought to note, however, that in adopting the Law of Non-Contradiction, we are perhaps not modeling the way people always think, because most of us, at least from time to time, harbor contradictory beliefs without realizing it. As soon as a contradiction is noticed, however, we usually take steps to eliminate it. The great dissatisfaction we feel upon encountering a contradiction seems to indicate something fundamental about how we carry on our reasoning. Just point out to someone that what they are saying involves a contradiction, and they will either remedy the conflict, or else try to show that no contradiction actually exists. In either case, there is a clear desire to avoid contradictions. There have been recent attempts to formalize the fact that our belief systems are not always consistent, and there are some non-symbolic systems which try to incorporate at least some kinds of contradictions, but we will not be able to deal with them in this book.

The Law of Excluded Middle has probably been the most suspicious of the three laws, principally because of its apparent support of the idea that the future is determined (or fated, or otherwise "already" set and unalterable). Consider first a statement about the past: "Foswell died yesterday." That sentence is true, or, if not true, then false. (We might not *know* which truth value the sentence has, but that is a different matter.) If it is a true sentence, then Foswell really is dead, and there's nothing we can do about it. Similarly, if the sentence is false, then Foswell really didn't die yesterday. (Maybe he died the day before; or maybe he's still alive; etc.) But in that case, too, no matter what we may do or wish now, the past is whatever it is: it is fixed, finished and unalterable.

Now consider a very similar claim about the future. Suppose someone today says, "Foswell will die tomorrow." The Law of Excluded Middle says that the sentence must be true, or, if not true, then false (there are no other possibilities). But if it really is true that Foswell will die tomorrow, and if, as the Law of Identity says, sentences may not change their truth values, then there appears nothing to do about the truth of "Foswell will die tomorrow"; nothing Foswell or anyone else does can change the truth of that sentence. So, too, if the sentence is false: If it is false that Foswell will die tomorrow, then no matter how hard Foswell tries to die tomorrow—jumping off a cliff, standing in front of an oncoming train, putting a loaded gun to his head and pulling the trigger—he will not die (tomorrow). Superman for a day! Of course, we might not *know* whether "Foswell will die tomorrow" is true or not. But again, that's not the point. The point, rather, is that the future will be whatever it will be, just as the past was whatever it was. If we accept the Law of Excluded Middle, then we must be prepared to admit that statements about the future already are true (or false, as the case may be).

But the thought that the future is just as unalterable as the past is too suspicious for many people to accept, and some writers have therefore suggested that there is after all a middle ground between *true* and *false*, at least for statements about the future. Even if, they argue, the past is fixed, the future is not; the future is something we can help create; our decisions have an effect on the future in a way they cannot have on the past. Such a third value for sentences might be something like "neither-true-nor-false-yet", or "indeterminate". And as long as a third value might be allowed, why not a fourth, and a fifth? Why not an infinite number of them? These and other issues are the subject matter for the study of *many-valued logics*. We, however, will be content to remain with our bivalent (two-valued) system, and we'll stay clear of the debate about whether the Law of Excluded Middle commits us to some kind of fatalism or determinism.

Since we are determined to have sentences (in our special sense) obey the three laws mentioned above, it will be useful to point out that often when we attempt to interpret the meanings of statements and arguments and to symbolize them with the tools which we will be developing, we will have to do a little "pushing" and "shoving". Consider, for example, the claim, "Foswell is dead". As it stands, the claim might be ambiguous if there is a Fred Foswell and a Fanny Foswell. Which one of them is supposed to be dead? The claim cannot be taken as a real sentence for our purposes unless it is such that it has one and only one truth value (either *true* or *false*); yet if Fred Foswell is dead and Fanny Foswell is not, then "Foswell is dead" is ambiguous. We are not allowed to say that the sentence is somehow *both* true and false, for then we would be disobeying the Law of Non-Contradiction. Furthermore, even if we do know which Foswell is meant, we will also have to know what time is being referred to. Perhaps Foswell was alive at 2 PM, but dead at 3 PM. We are not allowed to say that "Foswell is dead" changed its truth value in the course of that hour, for we would be running afoul of the Law of Identity. And we are not allowed to say that "Foswell is dead" is "sort of" true (or "sort of" false), because we would be disobeying the Law of Excluded Middle. In order to disambiguate the claim, we must specify exactly who the name "Foswell" represents, and exactly what time "is dead" refers to. Wherever necessary, we might revise the claim so that it is not ambiguous. Perhaps, for example, it would be enough to say, "Fanny Foswell is (at and after 3 PM, December 14, 1996) dead." In case there is more than one person named Fanny Foswell, we must find some way of saying which one we mean. And in case there is an ambiguity about the meaning of "is dead", we will have to find some means of specifying exactly what that means (no heart beat? no respiration? no lower brain activity?). And so on.

But it would place a great burden upon us to have to specify such details for all our sentences. Consequently, from now on we will simply assume that for each bona fide sentence, such details could be provided if required. But (lucky for us) they will seldom be required.

It would also be a great burden to have to repeat each sentence in its entirety, including all the details, each time we used it. For the sake of convenience, then, we will allow ourselves to abbreviate sentences by using single, capital letters: *A*, *B*, ..., *Z*, each of which, in a given context, will represent a different sentence which is either exactly true or, if not true, then exactly false.

1.2 Connectives (Operators)

Although any sentence whatsoever, no matter how complex, may be represented (abbreviated, symbolized) by a single letter (which we might as well call a ***sentence letter***), it sometimes happens that complex sentences are made up of simpler sentences joined together by ***connectives*** (sometimes called ***operators***) such as "and", "or", etc. Let's look at some of these connectives, and specify definitions and symbols for them.

AND

The sentence "John is clever and Mary is loquacious" may be symbolized by a single sentence letter; let's use *J*. But *J* is actually compounded of two simpler sentences: "John is clever" and "Mary is loquacious". If we let *J* stand not for the entire compound, but only for "John is clever", and let *M* stand for "Mary is loquacious", then we could write the compound as: *J and M*. But let's use instead a single symbol for the word "and", namely, the **ampersand**, "&". (Some logicians use "*", some use "•", some use "∧"; and there are others.) This gives us: *J&M*.

A compound sentence formed by the word "and" (or by the symbol "&") is called a

conjunction, and the sentences (or sentence letters) thereby ***conjoined*** are called ***conjuncts***. The word "and" in English has a variety of meanings, though, so in order to make our symbolization as precise as possible, we shall single out exactly one of those meanings. (We shall have to symbolize any other meanings of "and" in some other way. We'll talk more about that later on.) Here is a precise meaning for the ampersand:

> *A conjunction is true just in case both the conjuncts are true; otherwise it is false. (Alternatively: A conjunction is false if either or both of its conjuncts are false; otherwise it is true.)*

Thus, the compound sentence "John is clever and Mary is loquacious" is true just in case the sentence "John is clever" is true *and* the sentence "Mary is loquacious" is true. If either of those two sentences is false, then the whole compound sentence is false. Similarly, in our symbolic system, we may say that the compound sentence *J*&*M* is true just in case *J* represents a true sentence *and M* represents a true sentence.

Another (and, incidentally, more appropriate) way of defining the truth values which a conjunction can have is to give its ***truth table definition***, which sets out in tabular form the various possible truth values of the compound sentence based upon the various possible combinations of the truth values of the constituent sentences. The truth table for the conjunction *A*&*B* is:

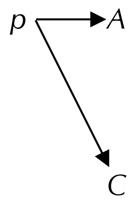
	the conjuncts		the conjunction
	A	B	A & B
1.	T	T	T
2.	T	F	F
3.	F	T	F
4.	F	F	F

A truth table is simply a list of all possible combinations of truth values for a given sentence or group of sentences (the left side of the table) and the resulting truth values for the compound sentence for each of those combinations (the right side). In the truth table above, there are exactly four possible combinations of Ts and Fs for two sentences. For each of those four possibilities (the four rows), the compound sentence must have one specific, exact, truth value. (More about how to construct truth tables in section 1.3 below.)

Notice that the truth table above is specifically for the conjunction *A*&*B*. How would we make a truth table for the conjunction *C*&*D*? Of course, it would be exactly the same as the above table, except that *C* and *D* would appear instead of *A* and *B*. Why? If we generalize a bit, we will readily see that we meant to express the truth table for "&" in such a way that it would apply to *any* two sentences connected by it. But *A* and *B* are not just any two sentences; they stand for (abbreviate) two sentences in particular (although we might not know which two). In order to express truth tables in a more general way, let us confuse things a little bit by introducing the lower case letters ***p***, ***q*** (and ***r*** and ***s*** and so on, as needed); such a lower case letter will stand for any sentence letter. That is, *p* will mean *any sentence (or sentence letter) at all*; and so will *q*. Specific sentences (or sentence letters) such as *A* or *B*, etc., are ***instances*** of the more general cases. Thus, *p*&*q* is a general pattern or schema which may be ***instantiated*** by substituting, say, *A* for *p* and *B* for *q* to get *A*&*B*. In this way we can express the definition of "&" more abstractly as:

	p	q	p & q
1.	T	T	T
2.	T	F	F
3.	F	T	F
4.	F	F	F

By the way, there is no requirement that the sentence which *p* stands for be a different sentence from the sentence which *q* stands for. Thus, *A*&*A* is a perfectly good instance of *p*&*q*. Nor is there any requirement that *p*, *q* and so on stand for simple (or ***atomic***) sentences. Since *A*&*B* is a particular sentence (we may call it a ***compound*** sentence, because it is not atomic), we may substitute it for *p*. And if we substitute, say, *C* for *q*, we end up with another instance of a conjunction, namely, (*A*&*B*)&*C,* where parentheses have been added in an obvious way. Although *p* need not represent a sentence different from *q*, we must stipulate that within a given context, *p*, *q*, and so on are not allowed to *shift* their meanings (or, in other words, they must obey the Law of Identity). Thus, if in the compound sentence schema (*p*&*q*)&*p*, we substitute *A* for the first *p*, then we must consistently substitute *A* for *p* wherever *p* occurs in that schema. (*A*&*B*)&*A* would therefore be OK, and so would (*A*&*A*)&*A* (where *A* is substituted for both *p* and *q*); but (*A*&*B*)&*C* would not be correct, because *p* would have to be representing ambiguously both *A* and *C* in the same context, and that is not allowed.



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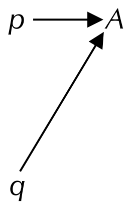


OK

OK

OR

Any two sentences connected by the word "or" are said to form a ***disjunction***, and the sentences thereby ***disjoined*** are called the ***disjuncts***. The English word "or" has at least two slightly different meanings: there is an ***inclusive OR*** and an ***exclusive OR***.

For the *inclusive* OR we use the symbol "∨" (some people use "+"), and we use this truth table definition:

	the disjuncts		the disjunction
	p	q	p ∨ q
1.	T	T	T
2.	T	F	T
3.	F	T	T
4.	F	F	F

The truth table may be summed up in this way:

> *An inclusive disjunction is false only when both disjuncts are false; otherwise it is true. (Alternatively: An inclusive disjunction is true when either or both of its disjuncts are true; otherwise it is false.)*

The symbolic formula *p*∨*q* may be translated: "Either *p* is true or *q* is true or both *p* and *q* are true." This is the sense of "or" (and sometimes "or else") used in sentences such as "The car went off the road because the road was slippery *or* because the driver fell asleep" (both might have been causes), and "Your dentist recommends that every night you either brush your teeth *or* floss" (doing both might be better still).

But there is another meaning for "or": "Either *p* is true or *q* is true, but not both". The previous meaning of "or" *included* the possibility of both p and q being true, while this second meaning *excludes* that possibility. This is the sense of "or" in such sentences as "When you buy a new car from us, you get a 500 dollar discount or a 500 dollar rebate" (surely they're not going to give you both!), and "I will get an A or a B in the course" (it's hard to imagine getting both grades). For this ***exclusive OR*** we will use the symbol "⊕". The truth table definition is:

	the disjuncts		the disjunction
	p	q	p ⊕ q
1.	T	T	F
2.	T	F	T
3.	F	T	T
4.	F	F	F

That truth table may be summed up this way:

> *An exclusive disjunction is true just in case both disjuncts have different truth values; otherwise it is false.*

Sometimes it is not easy to know whether "or" in an English sentence is being used in the inclusive or the exclusive sense. Suppose your friend tells you, "Either I'll be there by six o'clock, or I'll come over right after the game." I think you would be justified in taking this in the exclusive sense, under the assumption that the game will not be over until after six. But if you knew the game would be over much earlier, then perhaps the inclusive "or" would provide a better interpretation. Sometimes an extra phrase is added to a sentence to disambiguate the sense of "or". For example, "A and/or B" (or perhaps "A or B or both") might be used to draw attention to the inclusive use of "or"; and "A or B but not both" (or perhaps "One of A or B" or even "One of A and B") might be used to specify the exclusive sense. Very often, however, it won't make any difference whether the inclusive or the exclusive sense is used, as we'll see later on when we begin analyzing deductive arguments. As a rule of thumb, you may assume "or" to mean the inclusive "or", unless otherwise made explicit by some sort of qualifying phrase.

And notice that, as was mentioned earlier in the case of "&", the lower case letters p, q, etc. need not stand for atomic sentences, nor even for different sentences. Since, for example, $A\&B$ is a sentence, and since $A\&C$ is also a sentence, then connecting the two of them with "⊕" produces a new sentence: $(A\&B)\oplus(A\&C)$ is an exclusive disjunction of the general form $p\oplus q$. And so is $A\oplus(A\&A)$, where p is instantiated with A, and q is instantiated with $(A\&A)$. And so is $(A\vee B)\oplus(R\&S)$, where p is instantiated with $(A\vee B)$, and q is instantiated with $(R\&S)$. And so on.

NOT

The English word "not" simply reverses the truth value of the sentence on which it operates. "Not A" (or "A is not true") is the same as "A is false". And "A is not false" is the same as "A" (or "A is true"). We will symbolize "not" as "−" (a dash or minus sign). (Some people use a tilde, "~", some people use "¬", and some use a bar over the sentence letter.) The minus sign is more appropriately called the ***denial sign***. If you deny a sentence, then you are, in effect, affirming that it has the opposite truth value. (It has to be one or the other, according to the Law of the Excluded Middle, and since it isn't the one, then, according to the Law of Non-Contradiction, it has to be the other.) Here is the truth table:

	p	−p
1.	T	F
2.	F	T

That truth table is a precise way of representing the claim that:

> *If a sentence is true, then its denial is false; and if a sentence is false, then its denial is true.*

Exercise 1.1

* NOTE: For all exercises in this book, answers for starred problems are given in Appendix D.

Given that A is a false sentence, B is a true sentence, and C is a true sentence, determine the truth values for the following compound sentences.

* 1. $A \vee (B \mathbin{\&} -C)$
 2. $(A \mathbin{\&} -B) \vee C$
* 3. $B \mathbin{\&} (A \vee C)$
 4. $-[(A \vee B) \mathbin{\&} --C]$
* 5. $A \vee [B \mathbin{\&} -(C \mathbin{\&} A)]$
 6. $-[(A \mathbin{\&} --B) \vee -C]$
* 7. $-A \oplus (-B \vee C)$
 8. $A \oplus (B \oplus -C)$
* 9. $(A \mathbin{\&} B) \oplus -(A \mathbin{\&} B)$
 10. $-C \vee (B \mathbin{\&} -C)$

Given that "Alfonso goes to the party" is a false sentence, "Beryl goes to the party" is a true sentence, and "Clement goes to the party" is a true sentence, symbolize each of the following compound sentences and then determine their truth values.

 11. Alfonso goes to the party or Beryl goes to the party, but not both.
 12. Clement does not go to the party and neither does Beryl.
* 13. Neither Clement nor Beryl goes to the party.
 14. It's false that either Alfonso or Beryl goes to the party.
* 15. It's false that both Clement goes to the party and either or both of Beryl and Alfonso does not.

1.3 Truth Tables

Just about anything we could want to know about a sentence or a group of sentences can be determined from a truth table. To get a better understanding of how truth tables are constructed, we'll look at two examples.

Let's build a truth table for the sentence $(A \mathbin{\&} -B) \vee C$. We first note how many *different* atomic sentences there are, i.e., how many *different* sentence letters are used. In this case, there are three (*A*, *B* and *C*). Since each sentence has two possible values, three sentences together will have $2^3 = 8$ possible outcomes. You calculate these eight values of the compound sentence by determining the truth values of each of its components. Start with the simplest components, namely, the atomic sentences:

	A	B	C
1.	T	T	T
2.	T	T	F
3.	T	F	T
4.	T	F	F
5.	F	T	T
6.	F	T	F
7.	F	F	T
8.	F	F	F

Notice that the Ts and Fs have been arranged in a certain order, but the order is logically irrelevant; that is, any arrangement will do, just so long as all the combinations are there. To help ensure that all the combinations *are* there, you can follow this simple rule: For the right-most column, enter alternating Ts and Fs. For the next column to the left, alternate two Ts and two Fs. For the next column to the left, alternate four Ts and four Fs; then eight; then sixteen; and so on, doubling the numbers for each new column to the left.

The compound sentence we're examining, $(A\&-B)\vee C$, is a disjunction; that is, it is of the general form $p\vee q$. One of the disjuncts (the left one) is itself a compound sentence (a conjunction, in fact), and so before we can determine the eight values of the whole sentence, we need to know the eight values of that conjunction. While we already have the eight values of one of the conjuncts of that conjunction (namely, the sentence A, which is just the left-most column of the truth table above), we do not yet know the entries for the other conjunct, $-B$. Let's figure it out and enter it as a new column. Since $-B$ is the denial of B, each entry in the $-B$ column will simply be the denial of the corresponding entry in the B column.

	A	B	C	–B
1.	T	T	T	F
2.	T	T	F	F
3.	T	F	T	T
4.	T	F	F	T
5.	F	T	T	F
6.	F	T	F	F
7.	F	F	T	T
8.	F	F	F	T

We produced a column for $-B$ because it was a necessary step in determining a column for $A\&-B$, which, in turn, was a step toward getting a column for the whole sentence we're examining: $(A\&-B)\vee C$. So now we can calculate the eight values for the conjunction of A with $-B$, and enter the results as a new column. Notice how we do this: For each of the eight rows, we take the *value* (T or F) given for A in that row and the *value* given for $-B$ in that row and apply the truth table definition for "&" to get the result for that row. Thus, in the case of the first row above, A is true and $-B$ is false. Recall that the truth table definition for "&" says that when one sentence is true and the other is false, their conjunction is false. Here is the truth table for "&" again; p represents the A, and q represents the $-B$:

p	q	p & q	
T	T	T	
T	F	F	⇦
F	T	F	
F	F	F	

So we take the result—this F—and enter it for the first row under the new column of the truth table we are constructing. Row 2 of our new column has the same result. For row 3, *A* is true and −*B* is true, and according to the definition of "&", the conjunction is therefore true. And so on for the rest of the rows:

	A	B	C	−B	A & −B
1.	T	T	T	F	F
2.	T	T	F	F	F
3.	T	F	T	T	T
4.	T	F	F	T	T
5.	F	T	T	F	F
6.	F	T	F	F	F
7.	F	F	T	T	F
8.	F	F	F	T	F

We now have an entire column devoted to *A*&−*B*, which forms one of the disjuncts of our original sentence (*A*&−*B*)∨*C*. The other disjunct is the sentence *C*, the eight entries for which need no calculation, since they are already there as the third column in the truth table. So, we build a final column in the truth table by examining the eight values for the disjunction of the sentence (*A*&−*B*) with the sentence *C*:

	A	B	C	−B	A & −B	(A & −B) ∨ C
1.	T	T	T	F	F	T
2.	T	T	F	F	F	F
3.	T	F	T	T	T	T
4.	T	F	F	T	T	T
5.	F	T	T	F	F	T
6.	F	T	F	F	F	F
7.	F	F	T	T	F	T
8.	F	F	F	T	F	F

Let's work another example. Construct a truth table for (*A*∨*A*)&−*A*. As before, we first count the number of *different* sentence letters used. In this case there is only one, and so our truth table will have $2^1 = 2$ rows.

	A	
1.	T	
2.	F	

The sentence we're examining is a conjunction, and its possible values will depend upon the values of its constituent conjuncts, one of which is (*A*∨*A*) and the other of which is −*A*. Determining the two entries for *A*∨*A* is easy: For the first row (when *A* is true), we want to know the value of a true sentence in disjunction with itself—i.e., a true sentence in disjunction with a true sentence. The truth table definition for "∨" (see page 1–5) says that when one sentence of a disjunction is true and the other one is also true, then the disjunction as a whole is true. So *A*∨*A* gets T for row one. For the second row, *A* is false, and the truth table definition for "∨" tells us that a false sentence in disjunction with a false sentence yields a false disjunction:

	A	A∨A
1.	T	T
2.	F	F

The denial of *A* is easily added as another column:

	A	A∨A	−A
1.	T	T	F
2.	F	F	T

Since we're ultimately interested in the sentence $(A \lor A) \& -A$, we now use the truth table definition for "&" to calculate the final values of the conjunction, using the second column as the values of one of the conjuncts and the third column as values of the other conjunct:

	A	A ∨ A	−A	(A ∨ A) & −A
1.	T	T	F	F
2.	F	F	T	F

Exercise 1.2

* Answers for starred problems are given in Appendix D.

Construct complete truth tables for these sentences.

* 1. A ∨ (B & −C)
2. (A & −B) ∨ C
* 3. B & (A ∨ C)
4. −[(A ∨ B) & −−C]
* 5. A ∨ [B & −(C & A)]
6. −[(A & −−B) ∨ −C]
* 7. −A ⊕ (−B ∨ C)
8. A ⊕ (B ⊕ −C)
* 9. (A & B) ⊕ −(A & B)
10. −C ∨ (B & −C)

Exercise 1.3

* Answers for starred problems are given in Appendix D.

For each of the following pairs of sentences, show that each member of the pair must always have the same truth value as the other member. For example, in the first one below, you must demonstrate that whenever $A \lor B$ is true, $-(-A \& -B)$ is also true, and whenever $A \lor B$ is false, $-(-A \& -B)$ is also false. (Hint: Since in each case you want to compare the truth values of two sentences, you will have to use only one truth table for both sentences in a pair.)

* 1. A ∨ B, −(−A & −B)
2. A & (B ∨ C), (A & B) ∨ (A & C)
* 3. A, −−A
4. A & B, −(−A ∨ −B)
* 5. A ∨ (B & C), (A ∨ B) & (A ∨ C)
6. A, A ∨ A
* 7. A, A & A
8. −B ∨ (A & C), (A ∨ −B) & (−B ∨ C)
* 9. −(−A ∨ −B) ∨ C, (A ∨ C) & (B ∨ C)
10. (G & U) ∨ (G & −U), G

1.4 Consistency and Related Definitions

Of all the concepts we will be using, the idea of consistency is probably the most fundamental. A set (collection, group) of sentences is **consistent** if there is some valuation (i.e., some combination of truth values for the various sentences) which makes all the sentences in the set true at the same time. But if there is no such valuation, then the set of sentences is said to be **inconsistent**. For example, the set consisting of the three sentences

A ∨ B, −B, A & −B

is a consistent set, because there is *at least one way* that the three sentences could be true at the same time, as you can determine by means of a truth table:

	A	B	A∨B	−B	A&−B	
1.	T	T	T	F	F	
2.	T	F	T	T	T	⇦
3.	F	T	T	F	F	
4.	F	F	F	T	F	

Notice that the possibility of all three sentences' being true at the same time is exactly what row 2 in the truth table tells us. That is to say, the three sentences in the set will be true if the atomic sentence *A* is true and the atomic sentence *B* is false.

On the other hand the set

A∨B, −B, A&B

is an inconsistent set, because not all three sentences can ever be true at the same time. That is, if you construct the truth table, you will discover that there is no row that gives all three sentences the value T:

	A	B	A∨B	−B	A&B
1.	T	T	T	F	T
2.	T	F	T	T	F
3.	F	T	T	F	F
4.	F	F	F	T	F

The notion of consistency applies also to a single sentence (which we may think of as a set containing only one member). By definition, an inconsistent set of sentences is a collection of sentences which cannot all be true at the same time, so in the case of a set containing only one sentence, we get the claim that an inconsistent sentence is a sentence which cannot be true at the same time as the others in the set; but since there are no others in the set, we can say, more plainly, that an inconsistent sentence is simply a sentence which can never be true. An inconsistent sentence is also called ***contradictory***, ***self-contradictory***, ***logically false***, ***necessarily false***, or ***counter-tautologous***. All those terms mean the same thing, so use the one you like best. (What do you suppose the final column in the truth table of any inconsistent sentence will look like?) The sentence $A\&-A$, for example, can never be true, and so it is a contradiction. (Construct its truth table and examine it!) But the sentence $A\&-B$ can be true—that is, there is at least one way to make it true (namely, if A is true and B is false)—and so it is a consistent sentence, which is just another way of saying that if you construct its truth table, there will be at least one T under $A\&-B$.

A sentence which is true under every possible assignment of truth values is called ***tautologous*** (or ***necessarily true*** or ***logically true***). In other words, there is no way to make a tautology false. The sentence $A\vee-A$ is such a sentence. (What will the final column in the truth table for any tautologous sentence look like? Construct the truth table for $p\vee-p$ and find out!)

A sentence which is neither inconsistent nor tautologous is one which is neither false all the time (inconsistent) nor true all the time (tautologous); it must therefore be false some of the time and true some of the time. Such a sentence is called ***contingent***. (What will the final column in the truth table of a contingent sentence look like?) For example, the sentence $A\&B$ can be true (if A is true and B is true); but it can also be false (if either A or B is false). Construct the truth table for $A\&B$ and compare it with the truth tables for tautologies and contradictions.

If a sentence is either tautologous or contradictory (that is, either necessarily true or necessarily false—that is, not contingent), then it is said to be ***logically determinate***. That is, the laws of logic alone will suffice to determine whether the sentence is true or false. Otherwise, a sentence is ***logically indeterminate***. (And that means, you can see, that it is contingent.) Consider this sentence:

> **(1)** If the name of any city in England has more than two *n*s in it, then that name ends with an *n*.

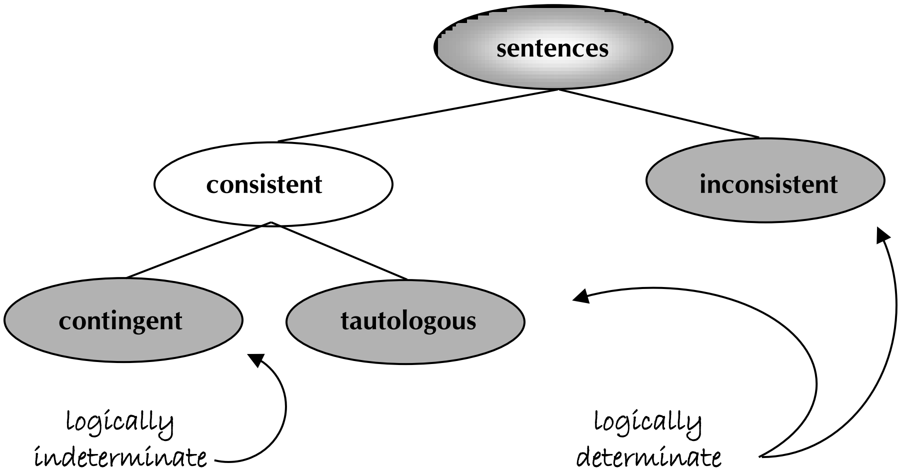
Is that a true sentence? Or false? I don't know, off hand. In order to find out, we would have to consult an atlas, or else we might travel all over England, paying attention to city names as we went. Consequently, we say that such a sentence is contingent (logically indeterminate), because it depends upon (is contingent upon) certain facts about the world, about which the laws of logic alone cannot give an unequivocal answer. (That is, its truth table will have some mixture of Ts and Fs in it.) But I do know that the following sentence is necessarily true, and hence is logically determinate:

> **(2)** The name of any city in England ends with an *n*, or else it doesn't.

I know that the sentence is true regardless of any matters of fact about geography or about the English manner of naming cities. Similarly, I know that the following sentence is necessarily false, and hence is also logically determinate:

> **(3)** If the name of any city in England has more than two *n*s in it, then that name does not have more than two *n*s in it.

I know that the sentence is false, just as certainly as I know that the previous sentence is true. Sentences (2) and (3) are logically determinate because their truth tables are unequivocal: the truth table for (2) has all Ts, and the truth table for (3) has all Fs.



Finally, two sentences are said to be ***logically equivalent*** if one is true whenever the other is true, and one is false whenever the other is false. Or, to put it another way, two sentences are logically equivalent if their final columns in the truth table are identical. For example, −(−*p*) is logically equivalent to *p*. (See Exercise 1.3.)

Exercise 1.4

* Answers for starred problems are given in Appendix D.

1. For each of the following sentences, determine whether it is a consistent sentence or an inconsistent sentence.

   a. (A & A) & −(B & B)
   * b. A & −A
   c. A ∨ −A
   * d. (A ∨ −B) & B

2. For each of the following sets of sentences, determine whether it is a consistent set or an inconsistent set.

   * a. A, B ∨ −A, −B
   b. −(A & B) ∨ C, −C, B
   * c. A, A ∨ −A, B, B ∨ −B
   d. (A & B) ∨ C, −C, −B

3. For each of the following sentences, determine whether it is tautologous, contingent, or contradictory.

   * a. A
   b. A ∨ (−B ∨ −A)
   * c. −[(A & B) ∨ (−A ∨ −B)]
   d. [(A ∨ B) & −A] ∨ [(A ∨ B) & A]

4. For each of the following pairs of sentences, determine whether the sentences in the pair are logically equivalent. (See Exercise 1.3.)

a. A, B
* b. C & –A, –(–C ∨ A)
c. A ∨ A, –(–A & –A)
d. A ∨ –A, B ∨ –B
* e. A & –B, B & –B

* 5. Are all tautologies logically equivalent to all other tautologies?

6. Are all contradictions logically equivalent to all other contradictions?

* 7. Is a conjunction consisting of a tautology and a contradiction logically determinate or logically indeterminate?

8. Is a disjunction consisting of a tautology and a contradiction logically determinate or logically indeterminate?

* 9. Are tautologies consistent with contradictions?

10. If a sentence is a contingent sentence, does that make it a consistent sentence?

1.5 Conditional Sentences

Compound sentences of the form “If..., then...” are called ***conditionals*** (or sometimes ***hypotheticals***). The sentence which follows the “if” is called the ***antecedent***, and the sentence which follows the “then” is called the ***consequent***. “If Adam is in town then Cory is in trouble” might be symbolized as $A \rightarrow C$, where the “→” is the “If..., then...” connective. (Some books use “⊃”.) Note that “consequent” is different from “consequence”. A conditional sentence is not necessarily the expression of a situation and its consequences. The relation between the antecedent and consequent is not necessarily one of cause and effect; nor is there any requirement that the antecedent represent something temporally prior to the consequent. Rather, the connection is *logical*—in terms merely of rules for establishing truth values. More on this later when we examine the nature of truth functionality.

Let us define the arrow (“→”) in this way:

	p	q	p → q
1.	T	T	T
2.	T	F	F
3.	F	T	T
4.	F	F	T

That truth table may be summarized:

> *A conditional is false only when the antecedent is true and the consequent is false; otherwise it is true.*

We’ll justify that truth table later on. For now, notice that “If..., then...” is not the only way to translate conditionals. Here are just seven of many ways to translate $A \rightarrow C$:

1. If *A*, then *C*.
2. If *A*, *C*.
3. *C*, if *A*.
4. In case *A*, *C*.
5. *A* only if *C*.
6. *A* provided that *C*.
7. Not *A* unless *C*.

The first is the standard or "canonical" translation. The second is the same as the first, except that the "then" has been omitted, simply as a matter of style. (The sentence "If you light a match in here, we'll all go up in smoke" is hardly different from "If you light a match in here, then we'll all go up in smoke.") The third version is also a trivial variation, where the consequent has been put first, probably as a matter of style or emphasis: "We'll all go up in smoke if you light a match in here." The fourth version is also a hypothetical: "In case *A* is true, then *C* is true"; for example, "In case Williams doesn't show up, Johnson will substitute for him". The fifth, sixth and seventh versions, along with some others, will be discussed later.

In order to get a better hold on why those translations above (and others) are acceptable, let's turn to the distinction between necessary and sufficient conditions. A ***necessary condition*** is something which is indispensable—a *sine qua non* (Latin for "without which not"). Being human, for example, is a necessary condition for being conscripted into the army. That is to say, if something is *not* human, then it cannot be conscripted into the army. On the other hand, being human is not a sufficient condition for being conscripted, because obviously not all humans are conscripted. A ***sufficient condition*** is one which, by itself, is enough to provide for the truth of something else. For example, being conscripted into the army is a sufficient condition for being human.

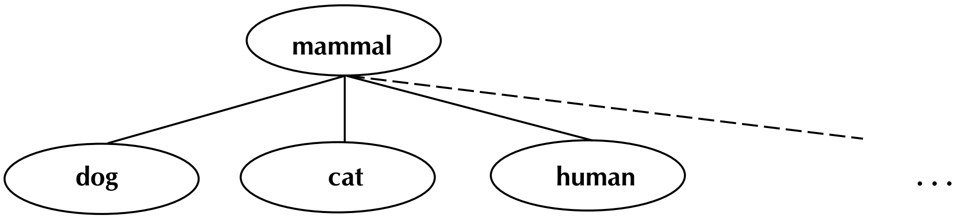
Exercise 1.5

* Answers for starred problems are given in Appendix D.

In each of the following pairs of concepts, one is usually understood as a sufficient condition for the other (and the other a necessary condition for the one). Determine which is which.

1. Mammal; dog.
2. Even number; divisible by 16.
* 3. Mother; relative.
4. Tautology; consistent.
5. Vehicle; automobile.

You can often determine the necessary/sufficient condition relations between two concepts by diagramming their relationships in terms of genus and species, especially if you add other species to the diagram. Consider, for example, the genus-species relationship between the category of mammal and the subcategories of kinds of mammals:



The concepts which appear lower in such a diagram are sufficient conditions for concepts higher up; and the higher up concepts are necessary conditions for the concepts appearing lower down. If something is a cat, then it is necessary that it be a mammal. On the other hand, a creature does not have to be a cat in order to be a mammal; being a dog (or human, or …) is also sufficient.

Now we are in a better position to justify the truth table for "→". An examination of the truth table for $p\to q$ reveals that q is a necessary condition for p (and therefore that p is a sufficient condition for q) for the following reasons: If the relationship $p\to q$ is true (i.e., rows 1, 3 and 4 in the truth table for "→"), then the truth of q is indispensable for (i.e., a necessary condition for) the truth of p, because just in case the truth of q is lacking (i.e., q is false, row 4), the truth of p is lacking as well. (Remember, row 2 does not count, because we are considering only those cases where $p\to q$ is true.) Similarly, whenever $p\to q$ is true, the truth of p is sufficient to guarantee the truth of q, because whenever p is true, q is also true (except in row 2, which, once again, we are not considering here).

Go over the preceding paragraph again and you will see that row 3 of the truth table has not been justified. Why have T in row 3 rather than F? The answer is that it must be T because it can't be F, and (by the Law of Excluded Middle) there are no other alternatives. Great! But why can't it be F? Well, if row 3 *were* F, then the truth table would look like this:

			?!
	p	q	p→q
1.	T	T	T
2.	T	F	F
3.	F	T	F
4.	F	F	T

Won't work!

But then $p\to q$ would have the same truth table as $q\to p$. (Check it out for yourself.) But if $p\to q$ and $q\to p$ had the same truth table, then they would be logically equivalent. (See the definition of logical equivalence above in Section 1.4.) But if they were logically equivalent, then they would mean the same thing—or at least they would have the same logical effects. And that would be an undesirable result, because clearly, "Being human is a necessary condition for being conscripted into the army" is not the same as "Being conscripted into the army is a necessary condition for being human". Similarly, "Being conscripted into the army is a sufficient condition for being human" is not the same as "Being human is a sufficient condition for being conscripted into the army". (If this discussion seems a bit confusing, it would be worthwhile to study it again.)

The conditional has still other expressions in English. Consider this sentence: "You may take the advanced logic course only if you have taken the introductory logic course." Let us symbolize "You may take the advanced logic course" by A, and symbolize "You have taken the introductory logic course" by I. Clearly, I is a necessary condition for A. (Correlatively, A must be a sufficient condition for I; that is, being permitted to take the advanced course must be a sufficient condition for having already taken the introductory course.) Since A and I are related in terms of necessary/sufficient conditions, they may be expressed as a conditional (where, remember, the antecedent is the sufficient condition, and the consequent is the necessary condition): $A\to I$. Thus, any sentence of the form "*p* only if *q*" may be symbolized as $p\to q$. The phrase "only if" signifies that a necessary condition is going to be given.

Consider the sentence, "You may not take the advanced course unless you have taken the introductory course". The word "unless" tells us that a necessary condition is about to be mentioned, without which the first item (taking the advanced course) is not possible (i.e., not true). We might put it this way: "If you have *not* taken the introductory course, then you may *not* take the advanced course". Clearly, that is a straightforward "if…then" sentence which can be symbolized as $-I \rightarrow -A$. As it turns out (you can check this with a truth table), the sentence $-I \rightarrow -A$ is logically equivalent to $A \rightarrow I$. So, sentences of the form "not *p* unless *q*" may be translated $p \rightarrow q$ (or as $-q \rightarrow -p$).

1.6 Some Words of Caution

If … then

Translations from English to symbolic notation can get messy. It is usually easier to translate from symbolic notation to English. But we should expect this, because English (and other natural languages) are ambiguous, shifty, context dependent, and full of emotion and innuendo; they usually convey much more information than we can hope to make explicit in any precise symbolism. Even an apparently straightforward sentence such as, "If you give me six dollars, I'll give you this book", is perhaps a bit misleadingly translated by means of the arrow (e.g., $S \rightarrow B$). Suppose, for example, you *don't* give me six dollars; then the implication is that I will not give you the book. But look back at the truth table for "→". Given that $S \rightarrow B$ is a legitimate promise which I make to you, i.e., that it is true (rows 1, 3 and 4 of the truth table), and then given that you don't give me six dollars (*S* is false; rows 3 and 4 of the truth table), then what may we say about *B*? According to the truth table, *B* might be either true or false without destroying the truth of $S \rightarrow B$. That is, I might give you the book even if you don't give me six dollars. In one sense, this is unexpected. But in another sense, this is quite all right, because there might be many different conditions under which I will give you the book, and receiving six dollars might be only one of them.

Not … unless

Similarly, suppose I say, "I won't give you the book unless you give me six dollars." Such a claim would naturally lead you to believe that if you did fork over the six dollars, then I *would* give you the book. But if we translate "not *B* unless *S*" as $B \rightarrow S$ (according to the pattern described in the previous section), then according to the truth table, *B* might be false even though *S* is true. In such cases, "not *p* unless *q*" might better be expanded to "not *p* unless *q*, and if *q* then *p*", which would be rendered symbolically as $(p \rightarrow q)\&(q \rightarrow p)$. But this, as we will see in the section on biconditionals below, would make *p* and *q* logically equivalent sentences—it would mean that your giving me six dollars would be logically the same as my giving you the book. Suppose that for some reason you owed me six dollars. You could pay me back with six one-dollar bills, or with a five and a one, or with twenty four quarters, etc., or else (so it would seem) you could pay me back by having *me* give *you* a book! And that certainly seems a bit odd.

And

Sometimes the word "and" does not mean "&", but rather "→", as in, "You strike that match in here and we'll all go up in smoke." And, as long as we're talking about "and", the word "but" (as well as "although", "however", and other words and phrases signifying that additional information is about to follow) may be translated as "&", although in doing so we leave out something of what we usually intend by the word. "Johnson will attend the party but Smith will not" could be symbolized as $J\&-S$, even though the sentence seems to mean something like "Johnson will attend the party and (contrary to what you might expect) Smith will not". We just don't have the tools necessary to symbolize mental attitudes (such as surprise or chagrin) which sentences can have in various contexts. Consequently, we'll simply leave such mental attitudes out of consideration altogether. The lesson to be learned here is that English is not mechanically translatable into our symbolism. A little care is required in order to understand subtle meanings. And a little pushing and shoving is necessary in order to make our symbolic forms work.

1.7 Biconditionals

Some things are both necessary and sufficient conditions for other things. If B is necessary for A, we write $A \rightarrow B$. If B is sufficient for A, we write $B \rightarrow A$. But suppose that B is both necessary *and* sufficient for A. How shall we symbolize that? Simple: just put the two sentences together in a conjunction: $(A \rightarrow B) \& (B \rightarrow A)$. For the sake of convenience, a new symbol is often used to represent that conjunction: "↔". A sentence whose main connective is this double arrow is called a ***biconditional***. The truth table definition for the double arrow has exactly the truth table for the conjunction above:

	p	q	p ↔ q
1.	T	T	T
2.	T	F	F
3.	F	T	F
4.	F	F	T

The truth table for the biconditional may be summarized:

> *A biconditional is true only when the two sentences have the same truth values.*

$B \rightarrow A$ may be read "A, if B" (or "if B, then A"). But it may also be read "B only if A" (see §1.5). The "only if" is different from a mere "if" and of course indicates a necessary condition. "I will pass the test only if I take it" says that taking the test is a necessary condition (but not sufficient!) for passing it. Similarly, $A \rightarrow B$ may be read as "B, if A" (or "A only if B"). Since the biconditional is really a conjunction of $A \rightarrow B$ and $B \rightarrow A$, the double arrow can be read as "A if, and only if, B". Of course, "B if, and only if, A" will do as well. That is, they have the same truth tables. Sometimes the phrase "if and only if" is abbreviated "iff".

What is the truth table for the denial of the biconditional? Here it is:

	p	q	– (p ↔ q)
1.	T	T	F
2.	T	F	T
3.	F	T	T
4.	F	F	F

That table says that the denial of the biconditional is true only when the two sentences have different truth values (rows 2 and 3). And notice, by the way, that this truth table is identical to the truth table for $p \oplus q$, which is to say that $p \oplus q$ and $-(p \leftrightarrow q)$ are logically equivalent (and hence so are $-(p \oplus q)$ and $p \leftrightarrow q$).

Exercise 1.6

* Answers for starred problems are given in Appendix D.

Determine whether each of the following sentences is contingent, tautologous, or contradictory. (Reminder: what does the final column in the truth table for any tautology look like? for a contradiction? for a contingent sentence?)

* 1. $[-S \leftrightarrow (P \,\&\, Q)] \vee S$
2. $(A \,\&\, B) \leftrightarrow -(A \,\&\, B)$
* 3. $(A \,\&\, B) \leftrightarrow -(-A \vee -B)$
4. $(A \leftrightarrow B) \leftrightarrow (B \leftrightarrow A)$
5. $A \leftrightarrow -(A \leftrightarrow \leftrightarrow A)$
6. $(L \rightarrow M) \leftrightarrow (M \rightarrow L)$
7. $(A \leftrightarrow A) \,\&\, (-A \leftrightarrow -A)$
* 8. $R \leftrightarrow (R \leftrightarrow R)$
9. $(C \rightarrow D) \leftrightarrow (-C \vee D)$
10. $U \leftrightarrow -U$

1.8 Truth Functionality

A connective is ***truth functional*** if the truth value of a compound sentence made with that connective is a function solely of the truth values of its components. All other connectives are ***non-truth functional***. Thus, the connective "∨" is truth functional because the truth value of a disjunction is defined entirely in terms of the truth values of the disjuncts. Similarly for "&". Only truth functional connectives can be given truth table definitions, and so all our connectives ("&", "∨", and so on) are certainly truth functional.

On the other hand, the connective "because" is not always truth functional. "Joe is happy because he passed the test" is a compound sentence using the connective "because" to connect the two subsentences "Joe is happy" and "He (Joe) passed the test". Let us invent an arbitrary symbol to represent that connective: "⇐". We may symbolize the above sentence in this way: $H \Leftarrow P$. Now suppose we attempt to give a truth table definition for "⇐".

	H	P	H ⇐ P
1.	T	T	?
2.	T	F	?
3.	F	T	?
4.	F	F	?

How shall we fill in the column under "⇐"? For the first row, we are to suppose that *H* is true and *P* is true, i.e., "Joe is happy" is true, and "Joe passed the test" is also true. What is the resulting truth value of $H \Leftarrow P$? Can we say that Joe is happy *because* he passed the test? Maybe. But then again maybe not. (Perhaps Joe has indeed passed the test, but perhaps the reason he is happy is that a rich uncle died and left him a small fortune.) So the entry under "⇐" might be either T or F, for all we can tell. In this case, then, knowing the truth value of *H* and knowing the truth value of *P* is not sufficient to provide the truth value of $H \Leftarrow P$. But truth tables must be unambiguous. Consequently, we cannot give a truth table definition for "because" (at least not when we interpret it causally), and so it is not a truth functional connective. (By the way, there is a perfectly good way to understand "because" in terms of arguments, where "because" signifies one or more reasons which are supposed to justify a conclusion. But we'll get to that in the next chapter.)

1.9 Disjunctive Normal Form

If sentences must be either true, or, if not true, then false (but not both), how many *different* combinations of truth values for two sentences can there be? We already know the answer: four. That is, for each of the binary (two-place) connectives, there are four rows in the truth table. Now, how many different columns of four rows each are there? That is, how many different truth table columns for binary connectives can there be? Answer: sixteen. Here they are:

p	q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
T	T	T	T	T	T	T	T	T	T	F	F	F	F	F	F	F	F
T	F	T	T	T	T	F	F	F	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F

There are sixteen possible, different, binary truth functional connectives, one for each of the numbered columns above. We have already met some of them. The second, for example, is "∨". And the eighth is "&". You will be able to pick out "→", "↔" and "⊕" as well. We could, if we wished, invent symbols for all the other columns. But that's not necessary; nor would it be convenient. (Besides, some of the columns do not seem to coincide with any single word or concept in English.) We will invent a few more symbols later on, but for now it is interesting to note that all the connectives can be reduced to compound sentences involving the connectives "&" and "∨" (along with the denial sign). For example, the third column is this:

	p	q	3
1	T	T	T
2	T	F	T
3	F	T	F
4	F	F	T

Although that does not represent any connective we know so far, we can easily write a sentence using "&", "∨" and "−" which will have that truth table. Consider: Row 1 says that the sentence is true just in case p and q are both true. (That is, in row 1 there is a T under p and a T under q.) Row 2 says that the sentence is true in case p is true and q is false. And row 4 says that the sentence is true if both p and q are false. In all other cases (namely, row 3) the sentence is not true. So what do we have for the truth conditions of the sentence? We have *row1* or *row2* or *row4*. Any of those three will produce a true sentence, given the above table. And we can symbolize that, sort of: *row1* ∨ *row2* ∨ *row4*. Now, what is *row1*? It is the condition in which both p and q are true, or, for short, $p\&q$. And *row2* is the condition in which p is true and q is false: $p\&-q$. And, finally, *row4* is the condition in which both p and q are false: $-p\&-q$. Consequently, we can say that the sentence is true if $(p\&q)\lor(p\&-q)\lor(-p\&-q)$ is true. Now, if you were to construct the truth table for this long sentence, it would be identical to the table above. Try it and see.

The general procedure for doing this with any sentence is simple: Construct the truth table for the sentence. Then write down the disjunction of the cases (rows) in which the sentence is true, where each of those cases is a conjunction of the values of *p* and *q* for that case. We will say that a sentence so created is in ***Disjunctive Normal Form***. To be in Disjunctive Normal Form a sentence must be represented as a disjunction of one or more conjunctions, i.e., it must have this general form: (&) ∨ (&)…. (This is also known as ***Sum of Products***, because if you represent "&" by "•" and "∨" by "+", as some logicians do, then you will have things that look like (*p* • *q*)+(*p* • −*q*)+(−*p* • −*q*); and if "•" represents multiplication and "+" represents addition, then that symbolic sentence will represent the addition—the summing—of products).

Notice that a contradiction is a sentence which has no *true* row in its truth table (column 16 above), and so there is no Disjunctive Normal Form expression for it.

1.10 Conjunctive Normal Form

If we focus attention on the rows of a truth table whose final values are *false* instead of *true*, we can construct a ***Conjunctive Normal Form*** expression (also known as ***Product of Sums***). Write down the conjunction of cases in which the sentence is false, where each of those cases is a *disjunction* of the *denied* values of *p* and *q* for that case. Here, for example, is column 14 from the big truth table earlier:

	p	q	14
1	T	T	F
2	T	F	F
3	F	T	T
4	F	F	F

Rows 1, 2 and 4 are the false rows, and so we form their conjunction: *row1* & *row2* & *row4*. Each of those conjuncts is to be a disjunction, formed by using the *opposite* truth values of those given in the table: (−*p*∨−q)&(−*p*∨*q*)&(*p*∨*q*). This compound sentence will have the truth table above.

Notice that a tautology is a sentence which has no *false* row in its truth table, and so there is no Conjunctive Normal Form expression for it.

We will return to the issue of Disjunctive and Conjunctive Normal Forms in Chapter Three (§3.12).

Exercise 1.7

* Answers for starred problems are given in Appendix D.

Translate these sentences into Disjunctive Normal Form.

* 1. B & (A ∨ B)
2. [(−A ∨ B) ∨ B] & −(B & A)
* 3. −(A ∨ B) → (B ↔ −A)
4. (U → R) → R
5. D & (−D → M)
* 6. D → E
7. −E → −D

8. (–A → –A) → Y
9. (–A → –A) → –Y
* 10. (–A ↔ –A) → –Y

Translate these sentences into Conjunctive Normal Form.

* 11. B ∨ (A & B)
12. (–A & B) & [B ∨ –(B ∨ A)]
* 13. (A → –A) ↔ W
14. N ↔ (A ∨ –N)
* 15. –(A & B)

Chapter 1 Test

1. Given that A is true, L is true, P is false and S is false, determine the truth values of these compound sentences.
   a. –L ↔ P
   b. [(S → –S) ∨ P] & (A → –A)
   c. [–(–S & L) ∨ (P → S)] ↔ [–L → (–L & L)]
   d. [(P → –L) & (S ∨ P)] → (–L ∨ –A)

2. For each of the following sentences, construct a truth table in order to determine whether it is contingent, tautologous or contradictory.
   a. [B ∨ (A & –C)] & –B
   b. [O & (L ∨ A)] & –[(O ∨ L) & (O ∨ A)]

3. Determine which one or more of the following categories apply to each of the sentences below. Categories: Consistent, Inconsistent, Tautologous, Contingent, Logically Determinate, Logically Indeterminate.
   a. A → A
   b. (A & –A) → B
   c. (A ∨ –A) ∨ (B ∨ –B)
   d. A → (A → A)

4. Give the Disjunctive Normal Form for each of these mystery sentences.

a.

A	B	?
T	T	T
T	F	T
F	T	F
F	F	T

b.

A	B	?
T	T	F
T	F	F
F	T	T
F	F	F

5. Give the Conjunctive Normal Form for each of the sentences in Problem 4 above.

6. For each of the following pairs of sentences, show that each member of the pair must always have the same truth value as the other member.
   a. Q ↔ N, –Q ↔ –N
   b. C → D, –D → –C

— 2 —

Arguments and Meta-Arguments

2.1 Induction, Deduction and Validity

An argument is a set of sentences, one of which (the conclusion) is supposed to follow from or be implied by the others (the premises). If the argument is expressed in English, then certain key words are usually present to tell us which sentences are being used as premises and which one is the conclusion. Terms such as “hence”, “therefore”, “thus”, “it follows that”, “so”, “as a result of which”, “consequently”, and so on, are fairly reliable indications that the conclusion is about to be announced. Very often, but not always, the conclusion will be the last sentence. But sometimes it is the very first sentence, in which case the premises might be prefaced by words such as “since”, “for the reasons that”, “because”, “the evidence is”, “for”, and so on. The order of the sentences in an argument, like the order of sentences in a conjunction, is not logically important; it is usually a matter of style or emphasis.

A ***deductive argument*** is one in which the premises are intended to give perfect and complete support for the conclusion, whereas in an ***inductive argument***, the premises are intended to establish only the likelihood of the conclusion. Consider the following argument:

> All of Bateson’s past novels have dealt with the conflict of values between cultures,
> so that will surely be the theme of his newest book.

The sentence, “that will surely be the theme of his newest book”, is the conclusion (indicated not only by the meaning of the claim as a whole, but also by the word “so”), the evidence or support for which is given in the sentence “All of Bateson’s past novels have dealt with the conflict of values between cultures”. We may take this as an inductive argument, because nothing in the premise requires or even suggests that Bateson is committed to directing all his work to a single theme. He might, for all we know, surprise us with a different kind of novel. So the premise of that argument makes the conclusion probably true—i.e., it might be a pretty good bet—but that's the most we can say. Although the conclusion of an inductive argument is not guaranteed to be true (on the basis of the premises), we ought to note that some inductive arguments provide better support for their conclusions than other inductive arguments do. If you flip a coin 50 times, and each time it comes up heads, then you have pretty good (but by no means conclusive) evidence that the coin is not a fair coin, and so you might reasonably conclude that it will come up heads the next time you flip it. But you have even better evidence (although still not absolutely conclusive evidence) that it is not a fair coin if you have flipped it 100,000 times and each time it has come up heads.

Now consider this argument:

> Smith won the tournament if she beat Jones in the final game. She did beat Jones in the final game. Therefore, she won the tournament.

The truth of the conclusion, "she won the tournament", is positively guaranteed if the premises are true. We might not *know* whether the premises are true; we might believe they are true and yet be mistaken. We might even believe they are false; and perhaps they are. Still, we say that the premises, *if* true, do give perfect support for the conclusion. And notice that we cannot say that sort of thing about the Bateson argument. There, too, we might not know whether the premises are true—whether, that is, all of Bateson's works really have dealt with the conflict of cultural values. But even if we did know, that would not be a guarantee that Bateson's next book would be on the same subject. Similarly for the coin tossings: We might be mistaken that the coin has come up heads on all 100,000 flips; but the argument is inductive not because of a possible failure of our knowledge, but rather because even if we did know for sure about the first 100,000 flips, we still would not be guaranteed a head on the very next flip.

If the premises of a deductive argument really would, if true, provide 100 percent support for the conclusion, then the argument is said to be ***valid***. Otherwise, the argument is ***invalid***. But what does "100 percent support", i.e., validity, really mean? A convenient way to define validity is in terms of invalidity:

> *A (deductive) argument is valid if and only if it is not invalid.*
>
> *A (deductive) argument is invalid if and only if there could be valuations for the premises and the conclusion such that the premises would be true and the conclusion false.*

Notice that these definitions have nothing to say about whether the premises and the conclusion actually *are* true or false. That is irrelevant to the issue of validity. The question to be asked is, "If the premises *were* true, could the conclusion possibly be false? I.e., is it logically possible that this argument could have true premises and a false conclusion?" If so, then the argument is invalid. If not, then the argument is valid. Thus, an argument whose premises are in fact false might nevertheless be a valid argument. And an argument whose premises are in fact true might nevertheless be an invalid argument. In order to determine the validity of an argument, we must examine all possible combinations of truth values of the sentences involved in the argument to see if there is some case where the premises are true yet the conclusion is false. The *presence* of even one such case indicates invalidity; and the complete *absence* of any such case indicates validity.

Here's an interesting question: What is the difference between an invalid deductive argument and any inductive argument? Both have the characteristic that even if their premises are true, the truth of the conclusion is not guaranteed. Is the argument above about Bateson's new book an inductive argument? Or is it an invalid deductive argument? We might say that it is both; but somehow that misses the point of most inductive arguments. We ought not to criticize the Bateson argument for failing to be deductively valid. Rather, we ought to judge it to be a more or less good (depending on the circumstances) inductive argument, because it does not seem to aspire to be a deductive argument at all. Similarly with the coin toss argument. Although it is deductively invalid to claim absolute certainty about the next coin toss on the basis of all those previous toss results, it is nevertheless inductively a pretty good bet. The difference between inductive arguments and deductive arguments, then, has something to do with what we intend the arguments to do, or what we expect from them. If an argument seems to promise certainty but doesn't deliver, or if we expect certainty, but don't get it, then we may say that the argument is an invalid deductive one. But if we are not looking for certainty, or if the argument seems to make no pretensions to certainty, then if we don't get it, we do not feel double-crossed, and we may say that the argument is inductive. Instead of calling inductive arguments valid or invalid, we call them good or bad, reasonable or unreasonable, probable or improbable, fairly certain or not very certain, and so on.

Exercise 2.1

* Answers to starred problems are given in Appendix D.

For each of the following arguments, identify the premises and the conclusion, and decide whether the argument should be understood as deductive or inductive.

* 1. On the basis of Pat's report, together with the treasurer's analysis of our spending for the past fiscal year, the committee feels that a rate increase would be acceptable to the membership.

2. Nobody other than Parker and Higgins had the security access codes. Furthermore, the security log shows a normal termination at 11:30 PM and normal restart at 11:45. Since normal terminations and restarts are possible only with the appropriate access codes, and since fifteen minutes is just enough time to get in, open the safe and get out again (provided you know what you're doing), and since Higgins has an air-tight alibi, it's easy to see that Parker is the thief.

* 3. Today the mouse will play. Why? Simple: When the cat's away, the mouse will play. And today the cat is away.

4. "What's the problem?" the drunk driver thinks to himself. "I haven't hit anything yet!"

* 5. If everything is predetermined, then there is no truly free choice. And if there is no truly free choice, then there is no place for praise and blame. But clearly not everything is predetermined, because praise and blame do indeed attach to people's actions.

6. Either we stop for dinner or we just starve as we continue driving until we're exhausted. So we're stopping for dinner, because there's no way we're going to continue driving.

7. I'm sure I'm going to hate Bellingham's new movie. I've seen all four of her other films, and they were just terrible.

8. It's a curious fact, but the stock market's ups and downs pretty much follow the rise and fall of the hemlines of ladies' dresses: when dresses are shorter, the stock market tends to rise, and when dresses are longer, the stock market falls. These days, skirts seem to be getting shorter, and so right now would probably be a pretty good time to invest in the stock market.

9. If you want to get a good job, you'll need a good education. If you want to retire early, you'll have to have a good job. Hence, if you want to retire early, you'll need a good education.

* 10. It appears that Johnson is a bit of a hypocrite. He's all the time telling everyone about the evils of eating meat, junk foods, fatty foods, fried foods, etc., but yesterday I caught him eating a hamburger and fries in a local fast food restaurant.

11. Helga lied about going to the party last week. She lied about having an exam this week. She misled her roommate about having paid the rent. I just don't trust her in anything at all.

12. Listen, Detective, I know you have your eye on Martin for the murder, but I've known Martin for maybe 10 or 12 years now, and I can tell you that he is just not the sort of person to murder someone.

13. This magnet will probably pick up that piece of dried spaghetti noodle, because it has picked up all sorts of small, thin objects, such as this needle, this bit of steel cable, this paper clip, and this nail file.

14. This magnet will pick up that piece of dried spaghetti noodle, because it has picked up all sorts of small, thin objects, such as this needle, this bit of steel cable, this paper clip, and this nail file; and whatever has picked up needles, cables and files will also pick up noodles.

15. This particular large, burned out circle in the corn field was certainly caused by an alien ship, because all large, burned out circles in corn fields are caused by alien ships.

16. If John is ill, then he won't be able to attend our meeting today. So he won't be able to attend our meeting today because he is in fact ill.

17. People who are ill do not, as a rule, attend meetings. John is ill, and so he will not attend the meeting.

18. Double 16 is 32, which is an even number. Double 17 is 34, which is also an even number. Etc. Therefore, the double of any number is an even number.

19. Finding Nemo is easy, because finding things under rocks is easy, and Nemo is under a rock.

20. If finding Nemo is easy, then he is under a rock. He is indeed under a rock. Therefore, Finding Nemo is easy.

From now on we will be concerned only with deductive arguments, and so we will judge them according to the definitions of deductive validity and invalidity given above.

2.2 Soundness

Besides validity and invalidity, there is another characteristic which deductive arguments can have, namely, soundness. A deductive argument, we know, may be valid even if the premises are not actually true. But if the premises of a valid deductive argument are, as a matter of fact, true, then the argument is said to be ***sound***.

Consider this argument:

> Charles Lindbergh is a hero, because if anyone is the first to fly non-stop from New York to Paris, then s/he is a hero, and Charles Lindbergh was the first to fly non-stop from New York to Paris.

The conclusion of that argument (“Charles Lindbergh is a hero”) follows deductively from the other sentences. (The word “because” in this case signals that the reasons—the premises—are about to be mentioned). It is a valid argument. Moreover, to the best of our knowledge, Lindbergh really was the first to make that flight; and it seems reasonable to say that such an accomplishment does make one a hero. If we are agreed about all that, then we are in the presence of a sound argument. Of course, it may turn out that Lindbergh was not, after all, the first; or we might wish to say that such a flight—bold and brave though it might be—is not sufficient to make one a hero. In either or both of those cases, we would deny the truth of one or more of the premises, and consequently we would deny that the argument is sound. We would, however, still have to admit that the argument is valid, because we recognize that *if* the premises were true, the conclusion would have to be true as well.

It is not usually a difficult matter to test the validity of deductive arguments: We need only see if there is any logically possible way that the premises could be true and yet the conclusion false; if there is such a way, then the argument is not valid. (We will be developing various tools and techniques to help us with that investigation.) But if we have determined that a particular argument is valid, how do we make the further test to determine whether the argument is sound? We would have to determine whether the premises are in fact true. In the case of the argument about Lindbergh, we would have to do some historical research. Perhaps, if we were seriously in doubt about the flight, we might examine Lindbergh’s plane, *The Spirit of Saint Louis*, to see if it was mechanically and aerodynamically capable of flying that distance—or flying at all. And as to whether or not we should call such a pilot a hero…. Well, we might want to analyze the concept of “hero” very closely. Suppose, for example, that Lindbergh had never even flown an airplane before; had misunderstood the nature of his own airplane and its capabilities; had failed to take into account wind and weather in his navigation; had grossly miscalculated the amount of fuel necessary for the flight; and had actually intended to fly to New Jersey. But suppose he ended up in Paris anyway. Suppose, that is, that everybody knew that Lindbergh was a raving maniac who flew a bucket of bolts to Paris just out of sheer, dumb luck. Probably “hero” would not be the appropriate word to describe him.

Establishing the truth of premises used in arguments can be a difficult task, and it is just the sort of thing which investigators in all fields of inquiry are engaged in. If we want historical evidence, we might turn to historians, archaeologists, anthropologists, and so on. For truths about airplanes, we might ask engineers in various fields of engineering. Logicians, however, are primarily concerned with the *relationships among sentences*, and not with the truth or falsity of isolated sentences (except for those truths and those falsehoods which can be known by logic alone, namely, the tautologies and the contradictions). Consequently, we will leave the determination of contingent (i.e., logically indeterminate) truths and falsehoods to the investigators in various fields, and devote our attention to questions about the validity of arguments but not their soundness.

2.3 Symbolizing Deductive Arguments

To symbolize a deductive argument, list the premises, separating them by commas. Then write the “turnstile”, “⊢”, and then the conclusion. Example:

↙ (This translates as “therefore” or “hence” or “it follows that”, etc.)

–U ∨ P, U ⊢ P

↑ ↗ ↖

premises conclusion

By the way, that example argument is valid. Why? Because it is not invalid. How can you tell that it is not invalid? Simple: It is impossible to assign truth values to the component atomic sentences (*U* and *P*) such that all the premises together would be true at the same time that the conclusion is false. You can construct a truth table to prove this:

	premise #2	conclusion		premise #1
	↓	↙		↓
	U	P	−U	−U ∨ P
1.	T	T	F	T
2.	T	F	F	F
3.	F	T	T	T
4.	F	F	T	T

On the left I have listed the two atomic sentence letters involved in this argument. (In fact, *P* by itself happens to be the conclusion.) Now remember that a truth table sets out all the logical possibilities, and in the truth table above it is impossible to find a row with both the premises true and the conclusion false (i.e., there is no row with T under premise #1 and T under premise #2 and F under the conclusion). But the possibility of true premises at the same time as a false conclusion is precisely the characteristic of an invalid argument. Since that can't happen in the argument above, it is not invalid. Hence, it is valid.

On the other hand, this argument is invalid: *M*→*Y*, −*M* ⊢ *Y*, because the truth table reveals a row which has T under both premises and F under the conclusion.

	M	Y	M → Y	−M	
1.	T	T	T	F	
2.	T	F	F	F	
3.	F	T	T	T	
4.	F	F	T	T	⇦

Exercise 2.2

* Answers to starred problems are given in Appendix D.

Use truth tables to test the following arguments for validity.

* 1. (A & B) ∨ −C, −B ⊢ C
2. A, B, C → −A ⊢ −C & B
3. A ⊢ (A → B) ∨ −(C & −F)
4. A ∨ (B & R) ⊢ (A ∨ B) & (A ∨ R)
* 5. −A, −Q, −F ⊢ −[A ∨ (−Q → F)]
6. C ↔ −Y, Y → E ⊢ C → −E
* 7. H ∨ L, L → (M ∨ C) ⊢ (−H & −M) → C
8. B, M ∨ (Q & A), Q → B ⊢ −M
* 9. E → (E → E) ⊢ A
10. (E → E) → E ⊢ A

Symbolize each of the following arguments, using the abbreviations (sentence letters) suggested. Then determine whether the arguments are valid or invalid.

11. If ***A***lfred is in town then ***B***rown is in trouble. Alfred is indeed in town. Therefore, Brown is in trouble.

12. Either ***A***lfred is a genius or ***B***rown is a fool (or both). But since Alfred is clearly a genius, it follows that Brown is a fool.

13. ***A***lcohol is flammable and so is ***b***enzene. Alcohol is not flammable. Therefore, benzene isn't.

*14. ***A***lcohol is flammable. Either ***b***unting on the third strike is ill-advised, or ***C***asey has some far out plan. Therefore, alcohol is flammable and Casey has some far out plan.

*15. If both ***L***awrence and ***M***argolis are convicted of the crime, then ***R***oberts will not be arrested. But Roberts is arrested. So Lawrence won't be convicted of the crime.

16. ***A***lfonso is a hero if and only if he acted **c**ourageously under dangerous circumstances. He did indeed so act. So he is a hero.

17. Since ***l***ogic is easy, either logic is easy or ***p***rime numbers greater than 1000 never end with the digit "3" (or both).

*18. Either ***l***ogic is easy or ***m***ath is hard (or both). You can see, then, that it is false that both logic is not easy and math is not hard.

*19. If ***t***his argument is valid, then the ***n***ext argument is also valid. But the next argument is not valid. Hence, this argument is not valid.

20. Plainly, the central idea of secession, is the essence of anarchy. A majority, held in restraint by constitutional checks and limitations, and always changing easily with deliberate changes of popular opinions and sentiments, is the only true sovereign of a free people. Whoever rejects it, does, of necessity, fly to anarchy or to despotism. Unanimity is impossible; the rule of a minority, as a permanent arrangement, is wholly inadmissible; so that, rejecting the majority principle, anarchy or despotism in some form is all that is left. (Abraham Lincoln, First Inaugural Address, 4 March 1861)

2.4 A Shortcut Method for Testing Validity

It is not always necessary to construct a truth table in order to test the validity of arguments. Attention to the definition of invalidity can lead us to a bit of a shortcut.

An invalid argument is one in which there is some possible valuation of the atomic sentences such that the premises come out true and the conclusion false. If that cannot happen, then the argument is valid. Is this argument invalid? $A{\to}B$, $-A \vdash -B$. If it is, then it is possible that the premises $A{\to}B$ and $-A$ are true while the conclusion $-B$ is false. We don't have to examine *all* possibilities for all assignments of truth values for all the sentences in the argument in order to see that the argument is invalid; we need only find *at least one way* to make the conclusion false and the premises true. (Only if there is *not* at least one such way is the argument valid.) That is, we needn't bother with those cases where the conclusion is true, or where either or both of the premises are false; such cases, if they are possible, actually give us no information at all about the validity or invalidity of the argument. That bears repeating: Whether the conclusion can be true, or whether the premises can be false, have no bearing on whether the argument is valid. We're interested only in the possibility of the conclusion's being false (while the premises are true), so let's see if it is possible that the conclusion could ever be false. (If the conclusion were a tautology, for example, then the argument would have to be valid, because there would be no way at all to make the premises true *and the conclusion false*, because there would be no way to make the conclusion false!) So: Is it possible, in the argument above, that $-B$ could be false? Clearly, it is possible: if B could ever be true, then $-B$ would be false in each such case. Well, is it possible that B could be true? Of course; any atomic sentence has just two possibilities, and one of them is true.

Let's reflect on what we've discovered so far. We're investigating this argument:

$$A \to B, -A \vdash -B$$

and we want to know if it is valid. We show that an argument is valid by showing that it is not invalid; and we determine whether or not the argument can be invalid by examining the possibility that the premises could be true and the conclusion false:

$$\underset{\text{true?}}{A \to B}, \ \underset{\text{true?}}{-A} \vdash \underset{\text{false?}}{-B}$$

The only way $-B$ could be false is if B were true, so let's mark B with a T.

$$\underset{\text{true?}}{A \to B}, \ \underset{\text{true?}}{-A} \vdash \underset{\underline{\text{false}}}{-\overset{\text{T}}{B}}$$

The Law of Identity requires that a sentence not change its truth value in a given context, so if, in the case we're trying to develop, B is to be true in the conclusion, then it must also be true in the first premise:

$$\underset{\text{true?}}{A \to \overset{\text{T}}{B}}, \ \underset{\text{true?}}{-A} \vdash \underset{\underline{\text{false}}}{-\overset{\text{T}}{B}}$$

We are looking for the possibility that the conclusion is false and the premises are true. We've just shown how the conclusion could be false. That's half the battle. Can we continue and show how the premises could be true (given that false conclusion)? The second premise, $-A$, could be true, namely, if A were false.

$$\underset{\text{true?}}{A \to \overset{\text{T}}{B}}, \ \underset{\underline{\text{true}}}{-\overset{\text{F}}{A}} \vdash \underset{\underline{\text{false}}}{-\overset{\text{T}}{B}}$$

Once again, a sentence must retain its value within a given context, so A in the first premise must be marked as F:

$$\underset{\underline{\text{true}}}{\overset{\text{F}}{A} \to \overset{\text{T}}{B}}, \ \underset{\underline{\text{true}}}{-\overset{\text{F}}{A}} \vdash \underset{\underline{\text{false}} = \text{INVALID}}{-\overset{\text{T}}{B}}$$

But look! That makes the first premise true. So now we have a possible valuation for the atomic sentences (A is F and B is T) such that both premises are true and the conclusion false. Since that is possible, then by definition it is an invalid argument.

All we've done here is, in effect, picked out certain rows in the truth table (and ignored others) as we went along. We ended up picking out the same information as is provided in row 3 of the following table:

	A	B	A → B	– A	– B	
1.	T	T	T	F	F	
2.	T	F	F	F	T	
3.	F	T	T	T	F	⇦
4.	F	F	T	T	T	

Let's try another. Test this argument: *A*→*B*, *B* ⊢ *A*. We want to know if the argument is valid—if there is no way to make all the premises true and the conclusion false. We needn't start by looking at the conclusion (although it is often the most obvious starting point). We can also wonder if we can make one or more of the premises true, and see where that leads us. Suppose we examine the first premise, *A*→*B*. Can it be true? Yes. How? In three ways: if *A* is true and *B* is true; or if *A* is false and *B* is true; or, finally, if both *A* and *B* are false. Let's pick the first way and see where it leads us. Make *A* true and *B* true throughout the argument:

$$\underset{\underline{\text{true}}}{\overset{T}{A} \rightarrow \overset{T}{B}},\ \underset{\underline{\text{true}}}{\overset{T}{B}} \vdash \underset{\underline{\text{f}\cancel{\text{al}}\text{se}}}{\overset{T}{A}}$$

Well, that assignment of truth values makes both premises true, all right; but it makes the conclusion true as well. But for an invalid argument, we have to show that the conclusion can be *false*. We have *not* been successful in making the premises true and the conclusion false. May we then say that the argument is not invalid and is therefore valid? No! That would be too hasty. Why? Because we have not shown that it is *impossible* that the premises be true and the conclusion false; we have shown only that you can't do it by assigning T to *A* and T to *B*; we haven't shown that there is no other way. We started by examining the first premise, *A*→*B*, and we chose *one* way to make it true. But since there are two other ways, let's see what happens with one of those other ways. How about when *A* is false and *B* is true?

$$\underset{\underline{\text{true}}}{\overset{F}{A} \rightarrow \overset{T}{B}},\ \underset{\underline{\text{true}}}{\overset{T}{B}} \vdash \underset{\underline{\text{false}}}{\overset{F}{A}} = \text{INVALID}$$

Now we have it. The argument is invalid.

The moral of the story is twofold: (1) Either prove that there is *at least one way* to make the premises true and the conclusion false, or else prove that there is *no way* to do so. (2) If you can avoid it, don't bother with compound sentences which can be evaluated in several different ways to produce the result you are looking for, because if you don't get the desired result one way, you'll have to go back and try the other ways; and that can get messy—so messy, sometimes, that it would have been easier to have constructed an entire truth table in the first place.

One more example: *A*→*B*, −*B* ⊢ −*A*. If the argument is invalid, then there will be some way to make all the premises true and the conclusion false. There are three ways to make the first premise true, as we saw in the last example, and so, as a matter of convenience, we'll skip that premise for now. There is only one way to make the second premise, −*B*, true; we have no choice there:

$$A \rightarrow B,\ \underset{\underline{\text{true}}}{-\overset{F}{B}} \vdash -A$$

If *B* is false anywhere, it must be false everywhere, so:

$$A \to \overset{F}{B},\ \underset{\underline{\text{true}}}{-\overset{F}{B}} \vdash -A$$

Notice that there is only one way to make the conclusion false, so let's do that:

$$A \to \overset{F}{B},\ \underset{\underline{\text{true}}}{-\overset{F}{B}} \vdash \underset{\underline{\text{false}}}{-\overset{T}{A}}$$

A, being assigned T in the conclusion, must now be assigned T throughout. But that forces the first premise to be false, and not true as we were trying to make it:

$$\underset{\cancel{\text{true}}}{\overset{T}{A} \to \overset{F}{B}},\ \underset{\underline{\text{true}}}{-\overset{F}{B}} \vdash \underset{\underline{\text{false}}}{-\overset{T}{A}}$$

All the assignments we made were forced, i.e., we did not pick among any options for a given sentence, and so we're stuck with no place to go. We may therefore conclude that it is not possible to assign truth values to the atomic sentences such that the premises come out true and the conclusion false. Hence, this is not an invalid argument; hence, it is a valid argument.

Exercise 2.3

Use the shortcut method to test the validity of the arguments in Exercise 2.2.

2.5 The Conditional Revisited

In section 1.5 we wondered if $p \to q$ is an adequate rendition of "if... then..." (and its variations). It is appropriate to raise still another caution with regard to conditionals. The following argument seems to be valid:

> If you manage to pass the final exam, then you'll get a passing grade in the course. You don't manage to pass the final. Therefore, you won't get a passing grade in the course.

But such an argument is symbolized as $M \to P$, $-M \vdash -P$, and it is a fallacy (i.e., invalid). Because it is a common fallacy, it has been given a name which is partly descriptive of its structure: the ***Fallacy of Denying the Antecedent***. Why is the argument invalid? You can use the truth table method or the short-cut method to show its invalidity. Or you can inspect the truth table for the conditional and notice that the first premise of the argument, $M \rightarrow P$, commits us to one of three lines in the truth table: line 1 or line 3 or line 4 (i.e., those lines where $M \rightarrow P$ is true).

	M	P	M → P	−M	−P
1.	T	T	T	F	F
2.	T	F	F	F	T
3.	F	T	T	T	F
4.	F	F	T	T	T

The second premise, $-M$, narrows that down to two lines, line 3 or line 4 (i.e., those lines where $-M$ is true). Now, does being committed to one of those two lines also commit us to the truth of the conclusion, i.e., to a line in which $-P$ is true? No. Line 4 does, but since we're committed to line 3 *or* line 4, we could legitimately go with line 3 (instead of line 4), which would make the conclusion false. Since it is *possible* that the premises could be true along with a false conclusion, the argument is invalid.

An argument very similar to the one above is this one:

> If you manage to pass the final exam, then you'll get a passing grade in the course. You do end up with a passing grade in the course. Therefore, evidently, you managed to pass the final exam.

But that argument is symbolized: $M \rightarrow P$, $P \vdash M$, and it, too, is a common fallacy, which has been named the ***Fallacy of Affirming the Consequent***.*

Why do those two fallacies seem, at first, to be valid? It is probably because we read more into the arguments than we represent in their symbolic versions. Specifically, it is probably because the sentence "If you manage to pass the final exam, then you'll get a passing grade in the course" is not, after all, accurately rendered as $M \rightarrow P$, because we quite naturally interpret the sentence against a background of other facts. Perhaps, for example, we imagine the circumstances in which such a promise about grades might be made: You are failing the course, and your one remaining hope for passing is to do at least minimally acceptable work on the final exam; furthermore, it is not to be expected that at this late date you could manage to get a perfect score on the final; or, even if you did, your other grades are so low that you would still only just barely pass the course. There are probably other facts we assume but do not make explicit: The instructor of the course cannot be bribed; he'll make no error in calculating the course grade; he won't suddenly take pity on you and give you an "A" for effort; and so on. All these facts, along with the "and so on", are probably *implicit* in your understanding of the meaning of the original conditional sentence. Under such an interpretation, the first premises of the arguments probably ought to be biconditionals, in which case both arguments can be shown to be valid. (Try them and see for yourself.)

Once again, the lesson to be learned is that translations from natural languages to artificial languages are bound to be less than perfect, because the artificial language, simply by being artificial, cannot hope to capture all of the background information which sentences in natural languages always implicitly carry with them. (Even "information" in that last sentence should not be taken as meaning precise propositions of the kind we are dealing with.) A very practical consequence of this is that research in computational schemes for "artificial intelligence"—i.e., attempts to program formal symbol manipulations which will exhibit some properties of intelligence—must usually be severely restricted to small, artificial domains, and cannot easily step out into the ordinary, lived world. (On

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* These two fallacies, by the way, ought not to be confused with two similar looking, but valid, argument forms, which we might call "Affirming the Antecedent" and "Denying the Consequent"—but which go by the more common names ***modus ponens*** and ***modus tollens***—and which may be symbolized as $p \rightarrow q, p \vdash q$ and $p \rightarrow q, -q \vdash -p$, respectively. We'll have more to say about them in Chapter 3.

the other hand, some wonderful and even useful success is possible by using brute force methods such as interconnected networks of faster and faster processors with access to huge stores of propositions and rules for manipulating those propositions.)

2.6 Metalogical Truths

Remember the truth table for "&"? And remember how it could easily be summarized by the following claim?

> *A conjunction is true just in case both the conjuncts are true; otherwise it is false.*

That summary is written in English; the truth table was written in our symbolic notation. When we talk *about* our symbolic notation, we are not talking *in* the symbolic language, but rather in a ***metalanguage***. A metalanguage is a language used to talk about some other language (which we may as well call the ***object language***). Some languages can be both metalanguage and object language at the same time, as when we use English to speak about the English language, as I'm doing in this very sentence. One good use of a metalanguage is to discuss certain general properties which an object language possesses. For example, it is a feature of the logic which we are developing here that if two sentences are logically equivalent (we may write this as $p \equiv q$), then they imply each other (i.e., if $p \equiv q$ is true, then both $p \vdash q$ and $q \vdash p$ are valid). We could invent a formal metalanguage for our symbolic notation which would allow us to give a rigorous proof of that claim. (That would make English the metalanguage for that metalanguage.) But that can get messy and confusing. Instead, we can give a non-rigorous, non-formal proof of the truth of the claim. Perhaps this will do:

> Whenever $p \equiv q$ is true, p and q have the same truth value (i.e., either both true or both false). But that means that both $p \vdash q$ and $q \vdash p$ would have to be valid, because the conditions for invalidity could not arise; that is, neither argument could have a true premise and false conclusion if the premise and conclusion had the *same* truth value. Q.E.D.

When necessary, or when desirable for illustration, we can include some schematic truth tables in such informal proofs. Consider this metalogical theorem:

> If the conclusion of a valid argument is a contradiction, then the premise(s) must be inconsistent.

Informal proof:

> Let the premise (or set of premises) be represented by *p* and the conclusion by *q*. Given that *q* is a contradiction, the truth table will have this general appearance:

p	q
.	F
.	F
T	F
.	F
.	F
F	F

> The number of rows in the table is not significant. All we are given to start with is that *q* is a contradiction, and so all of its rows must have F. We don't yet know anything about the premise, and so we must suppose that there could be one or more Ts and one or more Fs. We want now to prove that *p* is inconsistent, i.e., that the T cannot be there after all. And that's easy. If any Ts *were* there, then the argument $p \vdash q$ would be invalid. (T in the premise and F in the conclusion.) But we were told at the outset that it is a valid argument. So *p* can't have any Ts in its column, which means it has only Fs. And a sentence with all Fs in its column represents a contradiction.

A word of warning: Notice that in the above informal proof, everything was expressed in generalities. That is as it should be. It would not do, as a proof of the above metalogical claim, to use some *particular* premise and some *particular* contradictory conclusion. It would not do, for example, to put $A\&-A \vdash B\&-B$ on a truth table and point out that both of those sentences are contradictions. For what does that really show? It shows only that the *particular* argument $A\&-A \vdash B\&-B$ is a valid argument. But the original question asked about *any* valid argument with *any* premise and *any* contradiction as a conclusion. So we use *p*, *q*, etc. to represent *any* sentence; and we use a generalized truth table to represent any truth table of the required kind.

Exercise 2.4

* Answers to starred problems are given in Appendix D.

Give an informal proof of each of the following metalogical claims.

1. Any sentence whatsoever follows from (i.e., is a valid conclusion of) any inconsistent set of sentences.
2. Any tautology follows from (i.e., is a valid conclusion of) any set of sentences whatsoever.

* 3. If an argument is valid, then adding further premises of whatever kind will not turn that argument into an invalid argument.

4. If a countertautology follows from (i.e., is a valid conclusion of) a given set of sentences, then that set must be inconsistent.

* 5. Only tautologies follow from (i.e., are valid conclusions of) tautologies.

6. If *p* is some contingent sentence, and if *p* follows from (i.e., is a valid conclusion of) a set of sentences, *s*, then the set *s* might be contingent, or it might be inconsistent, but it could not possibly be a tautology.
7. If *p* is logically equivalent to *q* (i.e., $p \equiv q$), then $-p$ is logically equivalent to $-q$. (That is, if two sentences are logically equivalent, then so are their denials.)
8. If $p \vdash q$ is valid, then whenever *q* is false, *p* must be false.

* 9. If a sentence, *p*, follows from (i.e., is a valid conclusion of) a set of sentences, *s*, then the set of sentences consisting of *s* along with $-p$ must be inconsistent. (In other words, given any valid argument, the premises together with the denial of the conclusion will form an inconsistent set of sentences.)

10. If *s* is some set of sentences, and if *s* together with $-p$ is an inconsistent set of sentences, then $s \vdash p$ must be a valid argument.

* 11. If $s \vdash p$ is a valid argument, and if s is consistent, then the set consisting of s along with p is also consistent.

12. If both $s \vdash p$ and $p \vdash s$ are valid arguments, then $s \equiv p$ is true.

Note: *Pay special attention to 9 and 10, because the concepts involved there will become important when we study Indirect Proof and, even later, when we study the so-called "Tree Method".*

Chapter 2 Test

1. For each of the following arguments, identify the conclusion and the premise or premises. Make a case for the argument's being understood as deductive or inductive.
    a. That there is a Grand Watch-Maker can be made plausible in the following way. If you found a stone on the ground, you might think it was the product of ordinary natural and accidental forces. But if you found a watch on the ground, you would believe that there was some intelligent watch-maker. Now, there is much in nature that appears to be designed, just as a watch is designed. [This is a summary of William Paley's argument in his *Natural Theology* of 1802.]
    b. There must be simple substances, since there are compounds; for a compound is nothing but a collection of simple things. [Gottfried Wilhelm Leibniz, *Monadology*]
    c. Things which have nothing in common cannot be one the cause of the other. The knowledge of an effect depends on and involves the knowledge of a cause, so if two things have nothing in common, it follows that one cannot be apprehended by means of the other (because things which have nothing in common cannot be understood, the one by means of the other; the conception of one does not involve the conception of the other). *Q.E.D.* [Spinoza, *Ethics*, Part I, Axioms IV and V and Proposition III.]
    d. If you stay in a society when you have the freedom to leave, then it is reasonable to say that you have made an implicit contract to obey the laws of the society. Breaking a contract is unjust. Since I have lived in Athens all my life and was always free to leave, I cannot justly disobey the law which requires my death. [This is a summary of Socrates's views as expressed in Plato's *Crito*.]

2. Translate and test for validity:

    If Achilles is faster than a speeding arrow, and if Hector shoots an arrow toward a target, then Achilles will be able to reach the target before the arrow does. But Achilles will not be able to reach the target before the arrow does, because Hector doesn't shoot an arrow toward a target.

3. If an argument happens to have all true premises, is it a valid argument? Is it a sound argument?

4. Are these arguments valid?
    a. $\{[(A \to A) \to A] \to A\} \to A \quad \vdash \quad A$
    b. $S \to (N \vee -R),\ R,\ -N \quad \vdash \quad -S$

5. Show that if $s \vdash p$ is valid, then s, $-p$ is an inconsistent set.

— 3 —

The Method of Derivation

Sometimes it is not enough to prove that an argument is valid. Sometimes we want to know how a given set of premises can lead to a given conclusion. A proof of validity is a proof that a given conclusion follows from certain premises, but it does not provide us with a formula or recipe for deriving that conclusion from those premises. At other times we might be faced with a set of premises without knowing what conclusions follow from them. This is often the case with scientific (and other) investigations, where the question is of the form: Given the evidence, what conclusions may we draw? If the only tool we had to work with was a truth table method of proving validity, then we would have to scout around for likely looking conclusions and test each one of them in turn to see if it followed validly from the premises. It would certainly be more helpful if we had some sort of "generator" which could, on the basis of the given premises, churn out valid conclusions. Finally, even if we were to use the truth table method, it is (as you must agree by now) a real nuisance having to prepare a truth table for each proof. And imagine trying to prove the validity of an argument with more than just a few atomic sentences: for five atomic sentences, there would have to be 32 rows in the truth table; for six sentences, 64 rows; for seven sentences, 128 rows; and so on. Clearly, especially for complex arguments, some other method is needed. (As a very nice substitute for truth tables, we will be looking at the method of truth trees in Chapter 4. Truth trees, however, while being much simpler than truth tables, still do not show us the steps necessary to derive a conclusion from given premises.)

3.1 Some Properties of "$\vdash$"

Given that *s* is some set of (one or more) sentences, and *p*, *q* and *r* are single sentences, then the following *metalogical facts* about the relation denoted by "$\vdash$" are worth studying:

1. If $s \vdash p$ is a valid argument, then $s, q \vdash p$ is too. That is, the validity of an argument is not impaired by superfluous premises.
2. If $s, p \vdash q$ is valid, then $p, s \vdash q$ is valid. That is, the order in which the premises occur is of no logical significance.
3. If $s \vdash p$ and $p \vdash q$ are valid, then $s \vdash q$ is valid. That is, deducing q from s indirectly through p is an alternative to going directly from s to q.
4. If $s \vdash p$ and $s \vdash q$ are valid, then $s \vdash p \& q$ is valid (and *vice-versa*). That is, a premise, or set of premises, can have more than one sentence which is deducible from it.

Exercise 3.1

Provide an informal proof of the four properties of "⊢" mentioned above. (Note that the first one is the same as problem 3 in Exercise 2.4.)

Let's apply some of those properties to an example. We know (almost by inspection, but you can construct a truth table to make sure) that $p, q \vdash p \& q$ is a valid argument. Technically, we should call it an ***argument form*** (or type or template) which can have substitution instances (or instantiations or exemplifications) such as $A, B \vdash A \& B$ and $X, R \rightarrow \mathrm{M} \vdash X \& (R \rightarrow M)$, and so on. Now suppose we are faced with the following argument:

$$\mathrm{L} \lor \mathrm{Y},\ \mathrm{M} \leftrightarrow \mathrm{R} \ \vdash\ (\mathrm{L} \lor \mathrm{Y}) \ \&\ (\mathrm{M} \leftrightarrow \mathrm{R})$$

Is that a valid argument? The answer is, Yes; and we do not have to construct a truth table to prove it. Instead, we point out that the argument is an instance of the general argument form above, namely, $p, q \vdash p \& q$. Briefly put, any instance of a valid argument form must be a valid argument.

But now consider this argument:

$$\mathrm{L},\ \mathrm{M},\ \mathrm{O},\ \mathrm{P} \ \vdash\ (\mathrm{L} \,\&\, \mathrm{M}) \ \&\ (\mathrm{O} \,\&\, \mathrm{P})$$

We could say that on the basis of those premises, the slightly different conclusion $L \& M$ follows. That is, $L, M, O, P \vdash L \& M$ is valid, because it, too, is an instance of the general argument form mentioned above. (By the way, let's give that argument form a name, so that it will be easy to refer to from now on; it is called, appropriately, ***Conjunction***.) Notice that two of the premises (O and P) were not even used; but that's OK, because property (1) mentioned above for "⊢" says that unused premises do not interfere with validity. We can also say that $O \& P$ follows validly from those same premises, and for the same reason, namely, our new rule called Conjunction. Finally, property (4) above says that since both $L \& M$ and $O \& P$ follow from L, M, O, P, then so does their conjunction, $(L \& M) \& (O \& P)$. Thus we have shown, in a step by step manner, that $L, M, O, P \vdash (L \& M) \& (O \& P)$ is a valid argument. And we haven't had to use truth tables to do it.

Let's look at another example. Consider this argument form:

$$\mathrm{p} \rightarrow \mathrm{q} \ \vdash\ -\mathrm{q} \rightarrow -\mathrm{p}$$

We can be satisfied that this argument form (which, as we will see later, happens to be a version of something a bit stronger called ***Contraposition***) is valid. (Check it out with a truth table if you wish.) Since the *form* is valid, all its *instances* will be valid: $L \to M \vdash -M \to -L$ is valid; and so is $(S\&P) \to U \vdash -U \to -(S\&P)$; and so on. Now consider this argument:

$$A \to B, C, D \vdash (-B \to -A) \& (C \& D)$$

By property (1) of “⊢” we can disregard the second and third premises (C and D) and, on the basis of the first premise alone, validly conclude $-B \to -A$. Doing so is justified because it is an instance of the form Contraposition. Next, we can similarly disregard the first premise and validly conclude $C\&D$, which is justified by the former rule called Conjunction. Since both of those conclusions follow from the same set of premises, we may invoke property (4) of “⊢” and claim that $(-B \to -A)\&(C\&D)$ is deducible—or derivable—from the original premises.

What these examples show is that the properties of “⊢” allow us to “chain” together elementary arguments into longer and more complex arguments. The process of deriving a conclusion through a chain of simple arguments is called the ***Method of Deduction*** or the ***Method of Derivation***. If you were to examine hundreds or thousands of these kinds of examples, you would discover that certain simple argument forms are very handy for chaining together. Fortunately, you have been spared all that work. (There are other projects waiting for you.) Some of the simple argument forms which many logicians have found especially useful have been collected into the following list. Notice that they fall into two groups: the ***Elementary Argument Forms*** and the ***Equivalences***; the Equivalences can be used as argument forms, but they are something else besides, and we’ll investigate that further on. The list is important, because later on we will set down an inviolable rule for the method of derivation, namely, that every link in the chain *must* be an instance of either one of the Elementary Argument Forms or one of the Equivalence rules.

3.2 The Elementary Argument Forms

Absorption	$p \rightarrow (p \,\&\, q) \vdash p \rightarrow q$
Conjunction	$p, q \vdash p \,\&\, q$
Dilemma	$p \rightarrow q, r \rightarrow s, p \vee r \vdash q \vee s$
	$(p \rightarrow q) \,\&\, (r \rightarrow s), p \vee r \vdash q \vee s$
Disjunctive Syllogism	$p \vee q, -p \vdash q$
	$p \vee q, -q \vdash p$
Hypothetical Syllogism	$p \rightarrow q, q \rightarrow r \vdash p \rightarrow r$
Modus Ponens	$p \rightarrow q, p \vdash q$
Modus Tollens	$p \rightarrow q, -q \vdash -p$
Reduction to Absurdity	$p \rightarrow (q \,\&\, -q) \vdash -p$
	$p \rightarrow q, p \rightarrow -q \vdash -p$
Repetition	$p \vdash p$
Separation	$p \,\&\, q \vdash p$
	$p \,\&\, q \vdash q$
Weakening	$p \vdash p \vee q$

3.3 The Equivalences (Rules of Substitution)

Associative Laws .... $p \& (q \& r) \equiv (p \& q) \& r$
.... $p \vee (q \vee r) \equiv (p \vee q) \vee r$

Commutative Laws .... $p \& q \equiv q \& p$
.... $p \vee q \equiv q \vee p$

Contraposition .... $p \rightarrow q \equiv -q \rightarrow -p$

DeMorgan's Laws .... $-(p \& q) \equiv -p \vee -q$
.... $-(p \vee q) \equiv -p \& -q$

Distributive Laws .... $p \& (q \vee r) \equiv (p \& q) \vee (p \& r)$
.... $p \vee (q \& r) \equiv (p \vee q) \& (p \vee r)$

Double Negation .... $p \equiv --p$

Exportation .... $(p \& q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$

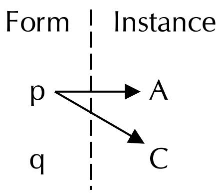
Idempotency .... $p \equiv p \& p$
.... $p \equiv p \vee p$

Material Equivalence .... $p \leftrightarrow q \equiv (p \rightarrow q) \& (q \rightarrow p)$
.... $p \leftrightarrow q \equiv (p \& q) \vee (-p \& -q)$

Material Implication .... $p \rightarrow q \equiv -p \vee q$

3.4 Some Comments on the Argument Forms

Notice that the elementary argument forms are expressed in terms of *p*, *q* and so on, to represent *any* sentences, simple or compound, in the same manner that was used in defining "&", "∨" and so on in section 1.2. There is no requirement that *p* must be a *different* sentence from *q*, but only that every *p* must represent the same sentence as every other *p* throughout a given context (e.g., a given argument form). Similarly for *q*, *r* and so on. Modus Ponens, for example, is given as *p*→*q*, *p* ⊢ *q*, and so *A*→*B*, *A* ⊢ *B* is a legitimate *instance* of Modus Ponens. But so is *A*→*A*, *A* ⊢ *A*, and so is (*A*&*C*)→*B*, *A*&*C* ⊢ *B*. But *A*→*B*, *C* ⊢ *B* would not be an instance of Modus Ponens, because it does not have the proper *form*: both *A* and *C* would have to share the role of *p* in the form (that is, *p* would *not* be representing the *same* sentence throughout the context), which is not allowed.



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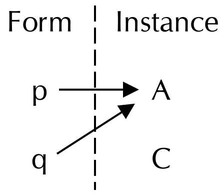


OK

Notice also that in the representations of the argument forms and the equivalences, −*p* means, quite simply, a sentence which is the denial of *p*; it does *not* mean that −*p* must represent a sentence which has a denial sign in front of it. If *p* represented, say, the sentence −(*R*&*W*), then −*p* would represent −−(*R*&*W*) (or, more simply, *R*&*W*). Consider the argument form Disjunctive Syllogism: *p*∨*q*, −*p* ⊢ *q*. A legitimate instance of this form would be this: −*A*∨*B*, *A* ⊢ *B*, where −*A* is an instance of *p*, and *B* is an instance of *q*, and therefore −*p* must represent the *denial* of −*A*, which is −−*A*, which is *A*.

Exercise 3.2

* Answers to starred problems are given in Appendix D.

1. Use either truth tables or the shortcut method to prove that each of the elementary argument forms is indeed valid.

For each of the following arguments, identify which of the elementary argument forms it is an instance of. For example, (*A*&*B*)→*R*, *A*&*B* ⊢ *R* is an instance of Modus Ponens.

2. (F ∨ Q) & (C → −R) ⊢ C → −R
* 3. {[M ∨ −(F → L)] & Y} → (B & −B) ⊢ −{[M ∨ −(F → L)] & Y}
4. L → M, C → −S, L ∨ C ⊢ M ∨ −S
5. (Q ↔ −B) → [T ∨ (W → Q)], −[T ∨ (W → Q)] ⊢ −(Q ↔ −B)
* 6. (Q ↔ −B) ∨ [T ∨ (W → Q)], −[T ∨ (W → Q)] ⊢ Q ↔ −B
7. −−S, B → (T ↔ S) ⊢ −−S & [B → (T ↔ S)]
8. (X & A) → (W ∨ B), (W ∨ B) → F ⊢ (X & A) → F
9. P & (P ↔ −W) ⊢ P & (P ↔ −W)
10. P & (P ↔ −W) ⊢ P ↔ −W

*11. $M \rightarrow (Q \;\&\; -R) \;\vdash\; [M \rightarrow (Q \;\&\; -R)] \vee \{[-T \leftrightarrow (-W \vee L)] \rightarrow -L\}$
12. $-F \;\vdash\; -F \vee C$
13. $(F \vee -F) \rightarrow (A \;\&\; -A),\; (A \;\&\; -A) \rightarrow U \;\vdash\; (F \vee -F) \rightarrow U$
14. $(B \;\&\; -B) \rightarrow U,\; (R \vee -R) \rightarrow (B \;\&\; -B) \;\vdash\; (R \vee -R) \rightarrow U$
*15. $C \vee -(A \rightarrow B),\; --(A \rightarrow B) \;\vdash\; C$
16. $C \rightarrow -(A \rightarrow B),\; --(A \rightarrow B) \;\vdash\; -C$
17. $(M \leftrightarrow -N) \rightarrow (R \rightarrow S),\; (M \leftrightarrow -N) \;\vdash\; R \rightarrow S$
18. $[(X \rightarrow -A) \vee (P \;\&\; -N)] \rightarrow [(U \leftrightarrow -S) \vee (P \vee Y)],\; [(X \rightarrow -A) \vee (P \;\&\; -N)] \;\vdash\; (U \leftrightarrow -S) \vee (P \vee Y)$
*19. $(A \;\&\; -U) \rightarrow [(A \;\&\; -U) \;\&\; (M \rightarrow M)] \;\vdash\; (A \;\&\; -U) \rightarrow (M \rightarrow M)$
20. $[A \rightarrow -(B \vee Q)] \vee (U \leftrightarrow A),\; -(U \leftrightarrow A) \;\vdash\; A \rightarrow -(B \vee Q)$

3.5 Some Comments on the Equivalences

The symbol "≡" represents logical equivalence. If $p \equiv q$, then p and q have identical truth tables. It is therefore related to "↔": if $p \equiv q$, then $p \leftrightarrow q$ is a tautology (and *vice-versa*).

The Equivalences (Rules of Substitution) are somewhat more powerful than the Elementary Argument Forms, because if $p \equiv q$ is true, then both $p \vdash q$ and $q \vdash p$ are valid (see problem 12 in Exercise 2.4), whereas with an Elementary Argument Form we are given only that $p \vdash q$ is valid, and we may *not* infer that $q \vdash p$ is also valid. For example, consider the Elementary Argument Form called Weakening: $p \vdash p \vee q$. You can see several things from the truth table. First, the column under p is not identical to the column under $p \vee q$; therefore, those two sentences are not logically equivalent. Second, the table shows that although $p \vdash p \vee q$ is valid, $p \vee q \vdash p$ is invalid (because there is a way to make $p \vee q$ true and p false at the same time):

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

In contrast, consider the Equivalence rule called Material Implication: $p \rightarrow q \equiv -p \vee q$. You can see in the truth table that not only do the two sentences have the same truth table, but also both $p \rightarrow q \vdash -p \vee q$ and $-p \vee q \vdash p \rightarrow \mathrm{q}$ are valid:

p	q	p → q	−p ∨ q
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

That is, all the Equivalences may be thought of (and used) as Elementary Argument Forms, and all the "reverse" versions of the Equivalences may also be used as Elementary Argument Forms.

But there is an additional use for the Equivalences, namely, they allow you to rewrite a sentence *or part of a sentence* using an equivalent expression. Suppose you have the following sentence to work with: $A \& (B \rightarrow C)$. Material Implication says that any sentence (even if it is a component of a more complex sentence) of the form $p \to q$ is *logically equivalent* to a sentence of the form $-p \vee q$; hence, the $B \to C$ part of $A \& (B \to C)$ may be rewritten as $-B \vee C$ to yield $A \& (-B \vee C)$. And the Equivalence rules go "both ways", so given $A \& (-B \vee C)$, Material Implication says that you may rewrite it as $A \& (B \to C)$.

The Equivalences, that is, allow you to substitute any expression for any equivalent expression (which is why the Equivalences are also called Substitution Rules), and such a substitution constitutes an inference (i.e., it acts as an argument form). Contrast that kind of power and versatility with the Elementary Argument Forms. Modus Ponens, for example, is $p \to q$, $p \vdash$ q. It says, "Given two (whole) premises whose forms are (1) a conditional, and (2) the antecedent of that conditional, you may validly infer a new (whole) sentence which is the consequent of that conditional". Modus Ponens, like all the other Elementary Argument Forms, gives us no warrant for going into a sentence and manipulating *parts* of it.

Exercise 3.3

* Answers to starred problems are given in Appendix D.

1. Use truth tables to prove that each of the equivalences is indeed an equivalence, i.e., that the sentences on either side of "≡" are logically equivalent.

For each of the following arguments, identify which of the Equivalences is being used. For example, −A&−B ⊢ −(A∨B) is a use of DeMorgan's Law.

* 2. (A ∨ −B) ↔ (M & A) ⊢ [(A ∨ −B) & (M & A)] ∨ [−(A ∨ −B) & −(M & A)]
3. [P & −(E ∨ −G)] → −(Y → P) ⊢ −−(Y → P) → −[P & −(E ∨ −G)]
4. −(−B & −−Q) ⊢ −−−(−B & −−Q)
* 5. (M & −B) ∨ [P & (M ∨ Y)] ⊢ [(M & −B) ∨ P] & [(M & −B) ∨ (M ∨ Y)]
6. (C & −C) ∨ [C & (C ∨ C)] ⊢ [(C & −C) ∨ C] & [(C & −C) ∨ (C ∨ C)]
* 7. L ∨ −(P & −X) ⊢ −[−L & −−(P & −X)]
8. (M & −B) ∨ [P & (M ∨ Y)] ⊢ (−B & M) ∨ [P & (M ∨ Y)]
9. −(L & −O) ⊢ −L ∨ −−O
*10. [(P → S) ∨ Q] & [(P → S) ∨ R] ⊢ (P → S) ∨ (Q & R)
11. (−S & T) → (I & −S) ⊢ −(S ∨ −T) → (I & −S)
12. −Q ∨ W ⊢ Q → W
13. (A ↔ B) ⊢ (A ↔ −−B)
14. (A ↔ B) ∨ (A ↔ B) ⊢ A ↔ B
*15. [(S & T) & −W] → B ⊢ (S & T) → (−W → B)
16. (S & T) → (−W → B) ⊢ [(S & T) & −W] → B
17. R ∨ L ⊢ −R → L
18. (F ∨ −Z) ∨ (F ∨ M) ⊢ (F ∨ −Z) ∨ (M ∨ F)
19. (F ∨ −Z) & (F ∨ M) ⊢ F ∨ (−Z & M)
20. (F ∨ −Z) ∨ (F ∨ M) ⊢ F ∨ [−Z ∨ (F ∨ M)]

3.6 Formal Proofs of Validity

In Section 3.1 we noted some properties of "$\vdash$" which allowed us to "chain" together a number of steps. We will adopt a convention to represent such a chain of arguments and to indicate the justification for each step (link) in the chain:

1. List the initial premises, one under another.
2. Draw a line under the list, and a line down the left side of the list.
3. To the left of the vertical line, number each of the premises.
4. Use the Elementary Argument Forms or Rules of Substitution (Equivalences) to generate the other links in the chain, and place these results under the initial list of premises.
5. Number each of those derived lines as you generate them.
6. To the right of each derivation—i.e., for each link—indicate which Elementary Argument Form or Equivalence was used in connection with which premise or premises to produce that new link.
7. Continue this process until the desired conclusion has been reached or until the cows come home.

Example #1: Prove $A \vee B, -B \,\&\, C \vdash A$
Proof (using the method of derivation):

1.	A ∨ B		initial premises
2.	− B & C	⊢ A	
3.	− B		from 2 by Separation
4.	A		from 1 and 3 by means of Disjunctive Syllogism

Note that $\vdash A$ was given off to the side in the example above. Strictly speaking, the $\vdash A$ is not a part of the proof; rather, it is merely a reminder that the derivation is to stop once A has been derived. Logically, however, the derivation can go on forever; each step after the premises constitutes a conclusion derived from the premises, and the only reason to stop deriving more conclusions is because of a prior decision to stop when some goal has been reached. Note also that each new line can be used to generate more lines, just as though it had been a part of the original set of premises. If $p \vdash r$ represents a valid argument, then you are allowed to add r to the set of premises, from which you may derive something further, such as s. That is, if $p \vdash r$ is valid, and if $p, r \vdash s$ is valid, then $p \vdash s$ is valid. If you're unsure about that, then examine this truth table:

	p	r	s	
1.	T	T	T	
2.	T	T	F	⇦ This row ruled out if p, r ⊢ s is valid.
3.	T	F	T	⇦ These 2 rows ruled out if
4.	T	F	F	⇦ p ⊢ r is valid.
5.	F	T	T	
6.	F	T	F	
7.	F	F	T	
8.	F	F	F	

The table shows that $p \vdash s$ must be valid once rows 2 and 4 have been ruled out.

Example #2: Prove $A\&B \vdash A \vee Y$
Proof:

1.	$A\&B$	$\vdash A \vee Y$	initial premise
2.	A		from 1 by Separation
3.	$A \vee Y$		from 2 by Weakening

Exercise 3.4

* Answers to starred problems are given in Appendix D.

Annotate (i.e., give the justifications for each line in) the following derivations.

* 1.

1.	$A \to B$	
2.	$C \to -B$	
3.	$--B \to -C$	from 2 by ________________
4.	$B \to -C$	from 3 by ________________
5.	$A \to -C$	from 1 and 4 by ________________

2.

1.	$(D \& E) \to F$	
2.	$(D \to F) \to G$	
3.	$(E \& D) \to F$	from 1 by ________________
4.	$E \to (D \to F)$	from 3 by ________________
5.	$E \to G$	from 4 and 2 by ________________

* 3.

1.	$(H \vee I) \to [J \& (K \& L)]$	
2.	I	
3.	$I \vee H$	from 2 by ________________
4.	$H \vee I$	from 3 by ________________
5.	$J \& (K \& L)$	from 1 and 4 by ________________
6.	$(J \& K) \& L$	from 5 by ________________
7.	$J \& K$	from 6 by ________________

4.

1.	$(M \vee N) \to (O \& P)$	
2.	$-O$	
3.	$-O \vee -P$	from 2 by ________________
4.	$-(O \& P)$	from 3 by ________________
5.	$-(M \vee N)$	from 1 and 4 by ________________
6.	$-M \& -N$	from 5 by ________________
7.	$-M$	from 6 by ________________

* 5.

1. $(Q \vee -R) \vee S$
2. $-Q \vee (R \& -Q)$
3. $(-Q \vee R) \& (-Q \vee -Q)$ from 2 by ______________________
4. $(-Q \vee -Q) \& (-Q \vee R)$ from 3 by ______________________
5. $-Q \vee -Q$ from 4 by ______________________
6. $-Q$ from 5 by ______________________
7. $Q \vee (-R \vee S)$ from 1 by ______________________
8. $-R \vee S$ from 7 and 6 by ________________
9. $R \rightarrow S$ from 8 by ______________________

6.

1. $T \& (U \vee V)$
2. $T \rightarrow [U \rightarrow (W \& X)]$
3. $(T \& V) \rightarrow -(W \vee X)$
4. $(T \& U) \rightarrow (W \& X)$ from 2 by ______________________
5. $(T \& V) \rightarrow (-W \& -X)$ from 3 by ______________________
6. $(T \& U) \vee (T \& V)$ from 1 by ______________________
7. $(W \& X) \vee (-W \& -X)$ from 4, 5, and 6 by ______________

* 7.

1. $L \leftrightarrow -Y$
2. $-Y \& B$
3. $(L \rightarrow -Y) \& (-Y \rightarrow L)$ from 1 by ______________________
4. $-Y \rightarrow L$ from 3 by ______________________
5. $-Y$ from 2 by ______________________
6. L from 4 and 5 by ________________

8.

1. $Q \rightarrow B$
2. $B \rightarrow U$
3. $U \rightarrow -B$
4. $Q \vee A$
5. $B \rightarrow -U$ from 3 by ______________________________
6. $-B$ from 2 and 5 by __________________________
7. $-Q$ from 1 and 6 by __________________________
8. A from 4 and 7 by __________________________

* 9.

1. $(L \vee U) \rightarrow [(L \vee U) \& C]$
2. $C \rightarrow H$
3. $-H \& R$
4. $(L \vee U) \rightarrow C$ from 1 by ______________________
5. $(L \vee U) \rightarrow H$ from 4 and 2 by ________________
6. $-H$ from 3 by ______________________
7. $-(L \vee U)$ from 5 and 6 by ________________
8. $-L \& -U$ from 7 by ______________________

10.
1. A → [(M & U) → E]
2. −E ∨ Q
3. −Q
4. U

5. [A & (M & U)] → E from 1 by ______________________
6. −E from 2 and 3 by ______________________
7. −[A & (M & U)] from 5 and 6 by ______________________
8. −A ∨ −(M & U) from 7 by ______________________
9. −A ∨ (−M ∨ −U) from 8 by ______________________
10. (−A ∨ −M) ∨ −U from 9 by ______________________
11. −A ∨ −M from 10 and 4 by ______________________

Exercise 3.5

* Answers to starred problems are given in Appendix D.

Give a proof of validity (i.e., use the method of derivation) for each of these arguments.

* 1. A → −B, C → B ⊢ A → −C
(Suggestion: Use Contraposition to set up a Hypothetical Syllogism.)

2. (G → −H) → I, −G ∨ −H ⊢ I
(Try transforming the antecedent of the conditional premise—using Material Implication—so that the result can be used with the second premise to generate the conclusion by means of Modus Ponens.)

3. D → (E ∨ F), −E & −F ⊢ −D
(Transform the second premise using DeMorgan's in order to set up Modus Tollens with the first premise.)

4. (J ∨ K) → −L, L ⊢ −J
(Try Modus Tollens and then DeMorgan's, and finally Separation.)

* 5. [(M & N) & O] → P, Q → [(O & M) & N] ⊢ −Q ∨ P
(Use Commutation and Association to transform the antecedent of the first premise, thereby setting up Hypothetical Syllogism. Then use Material Implication on the result.)

6. R ∨ (S & −T), (R ∨ S) → (U ∨ −T) ⊢ T → U
(Use Distribution on the first premise, then Separation. Use the result with the second premise, using Modus Ponens. Then transform the result using Commutation and then Material Implication.)

7. Q → X, B & −X, Q ∨ R ⊢ R
(Use Separation, Modus Tollens and Disjunctive Syllogism.)

* 8. A → L, −A → U, L → C, (U ∨ C) → B ⊢ B
(Use Material Implication on the first premise, then use the result along with the next two premises and Dilemma, setting up Modus Ponens with the last premise.)

9. $-(-B \vee -C),\ C \to R \ \vdash\ M \to R$
(Use DeMorgan's on the first premise, then Separation to set up a Modus Ponens. Use Weakening on the result, then Material Implication.)

*10. $-A \vee (M \,\&\, E),\ -E,\ A \vee (C \to -R) \ \vdash\ -(C \,\&\, R)$
(Use Distribution, then Separation, setting up Disjunctive Syllogism with the second premise. Use the result with the third premise and Disjunctive Syllogism again to get a sentence which can be transformed by Material Implication and DeMorgan's.)

11. $-B,\ -(L \,\&\, -Q) \to B,\ (E \to Y) \vee Q \ \vdash\ E \to Y$
(Use: Modus Tollens, Separation, and then Disjunctive Syllogism.)

12. $-(G \,\&\, -W) \to S,\ (H \to J) \vee W,\ -S \ \vdash\ H \to J$
(Use: Modus Tollens, Separation, and then Disjunctive Syllogism.)

13. $(S \to P) \vee O,\ -K,\ -(G \,\&\, -O) \to K \ \vdash\ -S \vee P$
(Use: Modus Tollens, Separation, Disjunctive Syllogism, and then Material Implication.)

14. $R,\ -(H \,\&\, -Z) \to -R,\ (W \to P) \vee Z \ \vdash\ -W \vee P$
(Use: Modus Tollens, Separation, Disjunctive Syllogism, and then Material Implication.)

15. $R,\ (-H \vee Z) \to -R,\ (W \to P) \vee Z \ \vdash\ -W \vee P$
(Use: Modus Tollens, DeMorgan's, Separation, Disjunctive Syllogism, and then Material Implication.)

16. $R,\ (-H \vee Z) \to -R,\ (W \to P) \vee Z \ \vdash\ -(W \,\&\, -P)$
(Use: Modus Tollens, DeMorgan's, Separation, Disjunctive Syllogism, Material Implication, then DeMorgan's.)

17. $R \,\&\, P,\ (-H \vee Z) \to -(R \,\&\, P),\ (W \to P) \vee Z \ \vdash\ -(W \,\&\, -P)$
(Use: Modus Tollens, DeMorgan's, Separation, Disjunctive Syllogism, Material Implication, then DeMorgan's.)

18. $R \,\&\, P,\ (-H \vee Z) \to -(R \,\&\, P),\ (W \to P) \vee Z \ \vdash\ -W \vee P$
(Use: Separation and then Weakening.)

19. $R \,\&\, P,\ (-H \vee Z) \to -(R \,\&\, P),\ (W \to P) \vee Z \ \vdash\ W \to P$
(Use: Separation, Weakening, then Material Implication.)

20. $-H \to (R \vee B),\ R \to (W \to P),\ -H,\ B \to N \ \vdash\ (W \to P) \vee N$
(Use: Modus Ponens and then Dilemma.)

3.7 Strategy

The method of derivation, like the similar method of proof in Euclidean geometry, presents an interesting challenge. We are faced with a set of strict rules (the Elementary Argument Forms and the

Equivalences), but no clear instruction on the *strategy* of using those rules. At each step of the derivation you may find perplexities. Should you use DeMorgan's Laws, or should you look for something to apply Hypothetical Syllogism to, or what? And where will it get you? How do you know when you're making progress toward the conclusion? Unfortunately, strategy is a very difficult art to communicate. It is a skill that, for the most part, you learn only because you use it. But if that's so, how can you learn it unless you already have it? And if you already have it, what's the need to learn it?

Fortunately, you already have at least some of the necessary skill. What rule would you use to prove this argument: A, $B \vdash A\&B$? No doubt you answered "Conjunction" immediately. So you do have some skill! Now you need only expand upon it, sharpen it, strengthen it. There are a few rules of thumb which might be of assistance.

1. Any parenthesized expression with a denial sign in front of it is a suspicious entity. Perhaps DeMorgan's Laws could be used.
2. Conditional sentences are usually very useful. Don't use Material Implication to transform conditionals into disjunctions unless you're pretty sure where you're going.
3. In many problems, the conclusion will obviously have to be obtained from one premise in particular. Pay constant attention to that premise and search for some other premise to work with it.
4. Try working backwards from the conclusion. See if you can imagine a sentence which, if you had it, you could use along with one of the existing premises to generate the conclusion. This imaginary premise will represent the start of a "wish list". Now examine that imaginary premise and pretend that it is the conclusion you need. Invent another imaginary premise which, along with the real premises, would get that new conclusion for you, and add it to the wish list. Keep going this way until you really can get to the most recent imaginary conclusion. Now start working your way toward the real conclusion through these imaginary premises, making wishes into reality, until the final conclusion is reached.

Be cautioned, however, that these rules of thumb are just that; they are by no means foolproof. And don't forget:

5. If all else fails, pick any Elementary Argument Form or Equivalence rule at random and see if it can be applied anywhere.

By the way, reworking problems which you have worked before is quite an excellent method of practice. When doing so, don't be afraid, if the going gets tough, to look at your previous answers. Part of the skill involved with the method of derivation is the seeing of *patterns*, and the more you work with these problems—especially the same problems—the more those patterns will start to leap out at you. Do the exercises in this section over and over. For some people, the method of derivation presents a fuzzy, almost impenetrable chaos of rules, until one day, somehow everything suddenly clicks into place. Be patient.

Exercise 3.6

* Answers to starred problems are given in Appendix D.

Translate each of the following arguments into symbolic notation. Then use the method of derivation to prove their validity. (By the way, they *are* all valid.) N.B.: Interpret "or" as inclusive, i.e., "∨".

* 1. If ***E***vans gets the job or his ***s***on goes to school, then both ***A***rnold and ***H***iggins will be relieved. ***I*** stand to gain if either Arnold or Higgins is relieved. Evans does get the job. Hence, I stand to gain.

2. If ***F***illbottom is convicted, ***H***asborough will resign. And if ***S***mythe-Jones doesn't resign, then if ***B***aby Face blows his top, ***M***agoo will be in hot water. Now, Smythe-Jones resigns or Fillbottom is convicted or Baby Face blows his top. But Smythe-Jones does not resign! So either Hasborough will resign or Magoo is in hot water.

* 3. If the ***f***irst disjunct of a disjunction is true, then the disjunction as a ***w***hole is true. Therefore, if both the first and the ***s***econd disjuncts of a disjunction are true, then the disjunction as a whole is true.

4. Since ***L***ouis refuses to cooperate, and since if Louis refuses to cooperate, we won't get a ***c***onviction, it follows necessarily that we won't get a conviction.

5. If ***A***lice drops out of the race, then ***C***harles will too. If ***B***randon drops out of the race, then so will ***D***oris. Either Alice or Brandon (or both) will drop out. Therefore, either Charles or Doris (or both) will drop out.

6. If ***A***lice drops out of the race, then so will ***C***harles. ***D***oris will drop out of the race if ***B***randon does. Either Charles won't drop out of the race or else Doris won't (or both won't). Hence, either Alice won't drop out of the race or Brandon won't drop out (or both).

* 7. If ***C***onnelly takes an early retirement, you can bet that both ***E***dwards and ***P***angborn will try to assume control of the company. But if Pangborn tries to assume control, we'll lose our contract with ***U***ptown Development. Now, since Connelly will indeed take an early retirement, it follows that we're going to lose our contract with Uptown Development.

8. Either or both of ***M***orris and ***P***erry have misjudged the mood of the voters. That can be proved from the following considerations: First, either those who vote for the proposed law are ***f***ools, or else both Morris and Perry have misjudged the mood of the voters. But, second, clearly those who vote for the proposed law are not fools.

* 9. If ***C***lark finds funding, the ***b***uilding will not be condemned, because if this building is condemned then the ***f***amilies presently living there will have no place to go. And if Clark finds funding, then the families presently living there will have a place to go.

10. ***A***rnold will succeed with his invention if and only if ***M***ayberry does not interfere. As a matter of fact, Mayberry does not interfere, and ***J***ohnson provides financial support. Hence, Arnold will succeed with his invention.

11. Either Quentin is prepared to run the test, or if Ellie runs the test, then Yergi will collect the results. If it's false both that Lawrence is prepared to run the test and that Quentin is not, then

Baker will run the test. But Baker will not run the test. Hence, if Ellie runs the test, then Yergi will collect the results.

12. If Adam orders pizza, Betty will not. If Charles orders pizza, then Betty will too. So if Adam orders pizza, then Charles will not.

13. Jeeves does not cook the goose, because if either Jeeves or Kent cooks the goose, then Lydia will not. But Lydia does cook the goose.

14. If D'Artagnan joins the fight, then either Eve or Flora (or both) will leave in disgust. Eve does not leave, and neither does Flora. So D'Artagnan does not join the fight.

15. If Target sells cell phones, then Sally will certainly buy her cell phone there. This is evident from the facts that either Robert reveals really rough routes to the roadside rutabaga region, or it is true both that Target doesn't sell cell phones and that William wants woven woolens. Also, if Robert reveals really rough routes to the roadside rutabaga region or William wants woven woolens, then either Sally will certainly buy her cell phone at Target or Target does not sell cell phones.

16. If the Queen abdicates, then there will be celebrations throughout the land. Either the Queen abdicates or Routledge goes into exile. Baxter resigns, but there will not be celebrations throughout the land. Hence, Routledge does not go into exile.

17. If Marigold marries Martin, Robert will be sad. This follows from the two facts (1) it's false that either Brubaker buckles or Carter does not understand what's going on, and (2) if Carter does understand what's going on, Robert will be sad.

18. If it is false both that Lawrence enters the race and Quentin does not, then Banquo enters the race. Either Quentin enters the race, or if Ellie wins, then Yosef does too. Hence, if Ellie wins, Yosef wins, because Banquo does not enter the race.

19. Either Williamsburg is not a friendly city or if Houston is, then Joplin is. If it's false that both Greensboro and Williamsburg are friendly cities, then Seattle is a friendly city. Hence if Houston is a friendly city, then so is Joplin, because Seattle is not.

20. Either Oakland is a friendly city or if Seattle is a friendly city, then Portland is too. If it is false both that Greensboro is a friendly city and that Oakland is not, then Knoxville is a friendly city. Alas, Knoxville is not a friendly city. Therefore, either Seattle is not friendly, or Portland is.

Exercise 3.7

* Answers to starred problems are given in Appendix D.

Use the method of derivation to construct a proof for each of the following.

1. $(-V \rightarrow W) \,\&\, (X \rightarrow W),\ -(-X \,\&\, V) \vdash W$

* 2. $G \vdash G \rightarrow [G \rightarrow (G \rightarrow G)]$

* 3. $[(Y \,\&\, Z) \rightarrow A] \,\&\, [(Y \,\&\, B) \rightarrow C],\ (B \vee Z) \,\&\, Y \vdash A \vee C$

4. $(M \rightarrow N) \,\&\, (O \rightarrow P),\ -N \vee -P,\ -(M \,\&\, O) \rightarrow Q \vdash Q$

5. [(A & B) & C] → R, −(−A ∨ −B) ⊢ −C ∨ R
6. C → R, L → Y, −L → C ⊢ Y ∨ R
* 7. −(S & M) → O, −M ⊢ O
8. {[P & (Q ↔ F)] ∨ −A} ∨ B, −B ⊢ [P & (Q ↔ F)] ∨ −A
9. −(R → S), F ∨ −R ⊢ −S & F
10. (L ∨ P) & −S, S ∨ −L ⊢ −P → Z
*11. −M, B, R → E, B → R, (E & A) → M ⊢ −A
12. U ↔ −C, −U ∨ M, −M & S ⊢ C
13. T → [U → (V → B)], −B, U & V ⊢ −T
14. (S & B) ∨ −W, S → (B → R) ⊢ W → R
15. (M & B) → −S, R → S, R ∨ D, −D, B ⊢ −M
*16. C → (R → A), −A, R ⊢ −C
*17. −D → −C, −B → −A, B → C, D → E ⊢ −(A & −E)
18. −M → −S, −(R & −S), −R → −A, M → L ⊢ A → L
19. (W ∨ Q) & −S, −S → −W ⊢ −Q → Z
20. (A → A) → B, B → (A → A) ⊢ A → A

3.8 Additions and Eliminations

The Elementary Argument Forms are merely convenient rules gathered together by people experienced in using the method of derivation. (As a matter of fact, some logicians put forward many rules in addition to the ones we have. Some logicians use fewer rules. And many of our rules go by other names in other books.) Since it is convenience which plays such a large role in the selection of rules, you may wish to add some of your own rules to the list. No license is required; no arduous training program need be taken. All you have to do is show that the rule you propose to use is a valid one, and you may prove it to be so by using the method of derivation itself, or by truth tables if you wish. For example, it might be handy to be able to go directly from a conjunction to a disjunction: $p \& q \vdash p \lor q$. We have no such rule, but we could stipulate one. Here is a proof of the rule's validity:

1.	p & q	
2.	p	1, Separation
3.	p ∨ q	2, Weakening

You may even give it a name, such as "*C* to *D*", or "Fred's Handy Rule" or anything else which strikes your fancy. Once proven valid, you may use the rule wherever you wish, just as though it had been on the original list of Elementary Argument Forms.

Some of the Elementary Argument Forms are not really necessary; their job can be performed by a judicious application of some other argument form or forms, perhaps with the aid of one or more of the Equivalence rules. For example, this is Modus Ponens: $p \rightarrow q, p \vdash q$. How else might q be inferred from those two premises, but without using Modus Ponens? Here is one of many ways:

1.	p → q	
2.	p	⊢ q
3.	−p ∨ q	1, M.I. (Material Implication)
4.	− −p	2, D.N. (Double Negation)
5.	q	3,4, D.S. (Disjunctive Syllogism)

This proof shows that given the argument form Disjunctive Syllogism, and given also the two equivalences Material Implication and Double Negation, then Modus Ponens is not really necessary.

Since rules can be eliminated without destroying the method of derivation, a natural question is, How many of those rules (and which ones) can be eliminated? Obviously, if we discarded *all* the rules, there would no longer be any method of derivation. But suppose we threw away all but one. Could we prove all valid arguments to be valid arguments if the only rule we could use was, say, Weakening? No. How about Weakening along with, say, Modus Ponens? Is there some minimal set of rules that could prove all valid conclusions?

There is a related question: How can we be sure whether the rules we have—or any set of rules at all, for that matter—are capable of proving all truth-table-provable conclusions? If a truth table shows that a particular argument is valid, will our method of derivation be able to prove it valid as well?

These and some other questions would take us too far afield, and so we will leave them unanswered here. We will, however, feel the need for some tools which are more powerful than what we have so far, and so we will soon develop three variations on the method of derivation.

Exercise 3.8

* Answers to starred problems are given in Appendix D.

* 1. Show that Modus Tollens is not needed, provided that all other argument forms and equivalences are available.

2. Similarly for Disjunctive Syllogism.

* 3. Similarly for Exportation.

4. Similarly for Contraposition.

3.9 Conditional Proof

Conditional Proof is another tool to be used in the method of derivation. Consider any valid argument at all. Let's call the set of its premises s and the conclusion q. We want to prove that $s \vdash q$. Now suppose we could prove that if we had an additional sentence (let's call it p), then, with the help of s (the original premises), we could derive q. Then we could represent that as: $s \vdash p \rightarrow q$. We're not claiming that the new sentence, p, is true; rather, we're claiming that *if* p were true, then (with the help of s) we could derive q. Notice that whether p is true or false does not affect the validity of the argument. That is, if $s \vdash \; q$ is valid, then so is $s \vdash \; p \rightarrow q$, as you can see from the general form that the truth table would have:

	s	p	q	p → q	
1.	T	T	T	T	
2.	T	T	F	F	⇦
3.	T	F	T	T	
4.	T	F	F	T	⇦
5.	F	T	T	T	
6.	F	T	F	F	
7.	F	F	T	T	
8.	F	F	F	T	

If $s \vdash q$ is valid, then these rows won't appear, which would mean that $s \vdash p \to q$ would also be valid (no row with T under s and F under $p \to q$).

Since it does not matter what *p* is, then it might be any sentence at all, simple or complex; it may be plucked out of thin air and have nothing to do with anything else in the original argument.

Swell. Interesting, even. But what good is it to be able to prove $s \vdash p \to q$, where *p* is any arbitrary sentence, if what you really want to prove is $s \vdash q$? After all, an examination of the truth table above reveals that $s \vdash p \to q$ might be valid even if $s \vdash q$ is not. (Look at row 4.) So even if we do prove that $s \vdash p \to q$ is valid, we certainly haven't proved that $s \vdash q$ is. So what's the point?

The answer is, we *don't* make the claim that if $s \vdash p \to q$ is valid, then so is $s \vdash q$; not at all. Rather, it's just that being able to derive such a conditional sentence can be very valuable. You can't prove everything using, say, DeMorgan's; it is nevertheless a valuable tool. And so it is with this new tool, Conditional Proof. The hard part is deciding what sentence to use as the new sentence, *p*.

Here is a typical case. Consider an argument form that already has a conditional sentence as its conclusion: $s \vdash r \to u$. In some cases it may be exceedingly difficult to use direct derivation to get that conclusion. Conditional Proof comes to the rescue. Simply pluck *r*, the antecedent of the conclusion, out of thin air, toss it into the argument, and mumble, "If *r* had been there at the beginning, what could I do with it?" Then start doing things with *r* and with the original premises. In particular, start doing things (that is, using the Argument Forms and Equivalences) such that *u* is the result. Once *u* turns up, the rule of Conditional Proof now allows you to announce that $r \to u$ follows from *s*.

In this case, all we've done is claimed that if $s, r \vdash u$ is valid, then so is $s \vdash r \to u$ (and *vice-versa*), which is justified by examining a truth table similar to the one above:

	s	r	u	r → u	
1.	T	T	T	T	
2.	T	T	F	F	⇦
3.	T	F	T	T	
4.	T	F	F	T	
5.	F	T	T	T	
6.	F	T	F	F	
7.	F	F	T	T	
8.	F	F	F	T	

If $s, r \vdash u$ is valid, then this row won't appear, which would mean that $s \vdash r \to u$ would also be valid (no row with T under s and F under $r \to u$).

For the sake of convenience, we will adopt a convention, the general form of which will indicate that an additional premise—we'll call it a *Conditional Assumption*—has been "conditionally" added to the original premises. And the result of the Conditional Proof will always be a conditional sentence. The Conditional Proof will look like a kind of proof-within-a-proof.

Example: Use conditional proof to prove:

$(A \vee B) \to (C \,\&\, D),\ (D \vee E) \to G \ \vdash\ A \to G$

Proof:

Step 1: Set up the proof as for ordinary derivation:

1.	(A ∨ B) → (C & D)	
2.	(D ∨ E) → G	⊢ A → G

Step 2: Invent a sentence to be used as a conditional assumption. Note, however, that it is only *conditionally* added, and so we ought to set it off from the main proof. In this example, we'll choose *A* as a conditional assumption, hoping to be able to derive *G*, at which point we can announce that *A→G* follows from the original premises. (But note that we could choose any sentence instead of *A*; logically, there are no restrictions whatsoever on what sentence we can use; we choose *A* for strictly strategic reasons.) Since we are, in effect, going to carry on a proof-within-a-proof, we will use roughly the same format for the inside (conditional) proof as we do for the outside (main) proof. That is, draw a vertical line, then draw a horizontal line to separate the (conditional) premise from whatever is derived from it.

1.	(A ∨ B) → (C & D)		
2.	(D ∨ E) → G		⊢ A → G
3.		A	Conditional Assumption

Step 3: The object of the conditional proof in this example is to show that the consequent, *G*, of the original conclusion can be derived from *A* (with the help of the original premises, as necessary). Note that the claim is not that *G* can be derived from the original premises alone. The claim, rather, is something like this: "Given the original premises, then if *A* were also given, then *G* would follow."

1.	(A ∨ B) → (C & D)		
2.	(D ∨ E) → G		⊢ A → G
3.		A	Conditional Assumption
4.		A ∨ B	3, Weakening
5.		C & D	1,4, M.P.
6.		D	5, Separation
7.		D ∨ E	6, Weakening
8.		G	2,7, M.P.

Step 4: But *the outside (main) proof is not completed until we have exited the conditional proof.* The justification for exiting the conditional proof will be called, simply, *Conditional Proof* (or *C.P.*), and instead of giving a particular line number as reference, we will indicate the range of line numbers used in the conditional block. In our example, we are saying that given the original premises, then if *A*, then *G*. That is, lines 3 through 8 show that *A→G* can be derived from the original premises.

1.	(A ∨ B) → (C & D)		
2.	(D ∨ E) → G		⊢ A → G
3.		A	Conditional Assumption
4.		A ∨ B	3, Weakening
5.		C & D	1,4, M.P.
6.		D	5, Separation
7.		D ∨ E	6, Weakening
8.		G	2,7, M.P.
9.	A → G		3 → 8, C.P.

Notice the all-important line 9, where we exit the conditional proof in order to return to the main proof. The range of line numbers constituting the conditional block is given as "3→8". The "→" is a helpful reminder of how line 9 is constructed. Look at line 9. It is simply *A* (which is the first line of the conditional block) → *G* (which is the last line of the conditional block).

Step 5: Continue the main proof as desired. In our example we needn't continue, because line 9 is the conclusion we were seeking.

Although the conditional proof above began on the first line after the premises and used, as the conditional assumption, the antecedent of the conclusion, it is not necessary to do things that way. A conditional proof may begin anywhere you wish. You are required only to exit the conditional proof before you claim that the "main" proof has been finished. This does not mean that the main proof is finished once the conditional proof is finished; you may continue the main proof if you wish. And you may subsequently begin still another conditional proof. Remember, Conditional Proof is a tool just like DeMorgan's, Modus Ponens, and so on, and so you use it when and where you think it might be helpful. Also remember that there is no restriction whatsoever on what sentence you may use as the conditional assumption.

One caution about the conditional proof: None of the lines *within* the conditional proof (for example, lines 3 through 8 in the proof above) are available *outside* of the conditional proof. That is to say, once the conditional proof is exited, all of the detail within that proof becomes invisible, goes away, is destroyed, becomes unusable.

Another example: Prove *A* ∨ *B*, −*A*, ⊢ *B* & [*R* → (−*R* → *Y*)].
Proof:

1.	A ∨ B		
2.	−A		⊢ B & [R → (−R → Y)]
3.	B		1,2, D.S.
4.		R	Cond. Assump.
5.		R ∨ Y	4, Weak.
6.		−R → Y	5, M.I.
7.	R → (−R → Y)		4→6, C.P.
8.	B & [R → (−R→Y)]		3, 7, Conj.

In step 4 of the proof above, the sentence *R* was used as the conditional assumption. Why *R*? Simply for strategic reasons. *R* was chosen for no other reason than that it seemed to offer a way to make some progress toward the conclusion. As with nearly every other aspect of the method of derivation, experience is the best guide.

Exercise 3.9

* Answers to starred problems are given in Appendix D.

Use conditional proof to prove the validity of these arguments.

* 1. A → (B ∨ C), B → D, C → D ⊢ A → D
2. (A & B) → (C ∨ D), A ⊢ B → (−D → C)
3. A → B, C → D ⊢ −(A ∨ C) ∨ (B ∨ D)
4. A → B, C → D ⊢ (A & C) → (B & D)
* 5. A → (−B ∨ C), D → (C ∨ E), −C ⊢ B → [−E → (−A & −D)] (Hint: Try using a conditional proof within a conditional proof.)

6. $G \vdash G \rightarrow [G \rightarrow (G \rightarrow G)]$
7. $(S \,\&\, B) \vee -W,\ S \rightarrow (B \rightarrow R) \vdash W \rightarrow R$
8. $(A \rightarrow A) \rightarrow B,\ B \rightarrow (A \rightarrow A) \vdash A \rightarrow A$
* 9. $C \rightarrow R,\ L \rightarrow Y,\ -L \rightarrow C \vdash Y \vee R$
10. $-M \rightarrow -S,\ -(R \,\&\, -S),\ R \rightarrow -A,\ M \rightarrow L \vdash A \rightarrow L$
*11. $C \vdash A \rightarrow A$
12. $A \rightarrow B,\ C \rightarrow -B \vdash A \rightarrow -C$
*13. $C \rightarrow M,\ -M \vee (X \,\&\, Y),\ -R \rightarrow -Y \vdash C \rightarrow R$
14. $L \rightarrow (U \rightarrow B),\ -(U \rightarrow B) \vee (S \,\&\, G),\ -W \rightarrow -G \vdash L \rightarrow W$
*15. $(D \,\&\, E) \vee -(H \leftrightarrow O),\ D \rightarrow (E \rightarrow P) \vdash (H \leftrightarrow O) \rightarrow P$
16. $[(M \,\&\, N) \,\&\, O] \rightarrow P,\ Q \rightarrow [(O \,\&\, M) \,\&\, N] \vdash Q \rightarrow P$
17. $A \rightarrow [(M \,\&\, U) \rightarrow E],\ -E \vee Q,\ -Q,\ U \vdash A \rightarrow -M$
18. $-S,\ -A \vee S,\ D,\ X \rightarrow [(Y \,\&\, D) \rightarrow A] \vdash X \rightarrow -Y$
19. $A \rightarrow (C \rightarrow M),\ (A \,\&\, C) \vee -W \vdash W \rightarrow M$
20. $Z \rightarrow (E \rightarrow L),\ (Z \,\&\, E) \vee -(W \,\&\, R) \vdash (W \,\&\, R) \rightarrow L$

3.10 Indirect Proof

If we want to prove that $s \vdash p$, it turns out that it is sometimes easier to prove that

(1) $s, -p \vdash q \,\&\, -q$

That is, we could prove that the original premises, along with the denial of the original conclusion, leads to some contradiction or other. Why is that helpful? Well, if argument (1) above is valid, then so is:

(2) $s \vdash -p \rightarrow (q \,\&\, -q)$

And, similarly, if (2) is valid, then so is (1). (If you don't buy this, review the previous section on Conditional Proof.) But now look at the elementary argument form called Reduction to Absurdity (or *Reductio* for short):

$$p \rightarrow (q \,\&\, -q) \vdash -p$$

This argument form says that any sentence which is a sufficient condition for a contradiction must be false. Or, in other words, if a contradiction is a necessary condition for the truth of a sentence, then that sentence *can't* be true—it must be false. (Recall that the relationship between a sufficient condition and a necessary condition when expressed as a conditional sentence is: *sufficient condition→necessary condition*.) We may apply the reductio to the conclusion of argument (2) above. If $-p \rightarrow (q \,\&\, -q)$ is true, then $-p$ must be false, which is to say that $--p$ must be true. Thus, if $s \vdash -p \rightarrow (q \,\&\, -q)$ and if $-p \rightarrow (q \,\&\, -q) \vdash --p$, then, by Hypothetical Syllogism, $s \vdash --p$. And since $--p$ is logically equivalent to p, then $s \vdash p$, and that is the very argument we wanted to prove in the first place.

The whole procedure may be summed up in this way: If the premises of an argument, together with the *denial* of the conclusion, result in a contradiction, then the denial of the denial of the conclusion (i.e., the conclusion itself) must follow validly from the original premises. This is called ***Indirect Proof*** (or sometimes ***Proof by Contradiction***) because, instead of directly deriving the conclusion from the premises, we show that *denying* the conclusion (given its premises) leads to a contradiction (an absurdity), and so, on the basis of those premises, the conclusion cannot be denied (i.e., cannot be said to be false when the premises are taken to be true).

Indirect Proof is used extensively in rigorous disciplines such as mathematics. But it is also found quite commonly in everyday discourse. Here, for example, are Pat and Chris:

> "I guess HomeMart is not going to have their annual sale", says Pat *[thereby expressing a conclusion].*
>
> "Why do you say that?" says Chris *[asking for the premises].*
>
> "Because we're on their mailing list *[a premise]*, and if they were going to have a sale *[the denial of the conclusion!]*, we would have received a notice in the mail by now *[an implication drawn from the premise along with the denial of the conclusion]*. But we haven't received a notice *[another premise]*."

Pat's reasoning is typical of Indirect Proof: If the denial of the conclusion *were* true, it would conflict with one or more of the premises, or else conflict with something implied by the premises. Or, to put it more generally, some contradiction would show up somewhere. Since denying the conclusion would get us into logical trouble, we ought *not* to deny the conclusion (i.e., we ought to affirm it).

Since the method of indirect proof relies on the method of conditional proof (plus Reduction to Absurdity), the layout for indirect proof will not be different. Here is the general form which an indirect proof will take. In this general form, we are deriving some conclusion, *p*, from some set of sentences, *s*:

1.	s		
2.		−p	Cond. Assump. (In this case, the *denial* of conclusion)
.		.	
.		.	*Regular derivation stuff.*
.		.	
n		q& − q	*Some contradiction or other.*
n+1	−p→(q& − q)		2 → *n*, Cond. Proof.
n+2	− −p		*n*+1, Reductio
n+3	p		*n*+2, D.N.

Example: Use Indirect Proof to prove: $A \vee (B \& C),\ -A \vee B \ \vdash\ B$.
Proof:

1.	A ∨ (B & C)		
2.	−A ∨ B		⊢ B
3.		−B	Cond. Assump. (denial of conclusion)
4.		−A	2,3, D.S.
5.		B & C	1,4, D.S.
6.		B	5, Separation
7.		B & −B	6,3, Conj.
8.	−B → (B & −B)		3 → 7, C.P.
9.	− −B		8, Reductio
10.	B		9, D.N.

By the way, the use of Double Negation is usually rather obvious, and so for the sake of convenience, it is often omitted. Thus, steps 9 and 10 above could have been combined into one without any loss of clarity. Strictly speaking, the proof is not complete without Double Negation, so I included it just to be picky. The same might be said of Commutation and Association; once you have used them a few times, they become very obvious steps, and you can then choose to omit them when there is no chance of confusion. (On the other hand, you can't go wrong by putting them in wherever they belong.)

Exercise 3.10

* Answers to starred problems are given in Appendix D.

Prove each of the following, using Indirect Proof.

* 1. $A \to B,\ C \to D,\ (B \vee D) \to -E,\ F \,\&\, E \ \vdash\ -A \,\&\, -C$
2. $(A \vee B) \to (C \to -D),\ (D \vee E) \to (A \,\&\, C) \ \vdash\ -D$
3. $A \to (B \to C),\ -C,\ B \to A \ \vdash\ -B$
4. $(A \vee B) \to (C \,\&\, D),\ (C \vee E) \to -N,\ N \to A \ \vdash\ -N$
* 5. $(M \to N) \,\&\, (O \to P),\ -N \vee -P,\ -(M \,\&\, O) \to B \ \vdash\ B$
6. $(-V \to W) \,\&\, (X \to W),\ -(-X \,\&\, V) \ \vdash\ W$
7. $(A \to A) \to B,\ B \to (A \to A) \ \vdash\ A \to A$
8. $(M \to N) \,\&\, (O \to P),\ -N \vee -P,\ -(M \,\&\, O) \to Q \ \vdash\ Q$
9. $-(S \,\&\, M) \to O,\ -M \ \vdash\ O$
* 10. $-M,\ B,\ R \to Q,\ B \to R,\ (Q \,\&\, A) \to M \ \vdash\ -A$
11. $A \to (B \vee C),\ B \to D,\ C \to D \ \vdash\ A \to D$
12. $(S \,\&\, B) \vee -W,\ S \to (B \to R) \ \vdash\ W \to R$
* 13. $A \to B,\ C \to D \ \vdash\ -(A \vee C) \vee (B \vee D)$
14. $A \to (-B \vee C),\ D \to (C \vee E),\ -C \ \vdash\ B \to [-E \to (-A \,\&\, -D)]$
15. $A \ \vdash\ A$
16. $A \to -B,\ C \to B \ \vdash\ A \to -C$
17. $(G \to -H) \to K,\ -G \vee -H \ \vdash\ K$
18. $D \to (E \vee F),\ -E \,\&\, -F \ \vdash\ -D$
19. $-W \to (R \vee S),\ -R \,\&\, -S \ \vdash\ W$
20. $-D \to -C,\ -B \to -A,\ B \to C,\ D \to E \ \vdash\ -(A \,\&\, -E)$

Indirect proof is not always the easiest way to go. In the answer to problem 5 above, for example (see Appendix D), the conclusion, *B*, was derived at step 8 without the need for introducing the denial of the conclusion and setting up the indirect proof. (Notice, however, that the proof cannot simply be stopped at line 8, because at that point we are still inside a conditional block, and conditional blocks *must* be exited before a proof is complete. But exiting the conditional block at that point would yield this line 9: $-B \to B$. Fortunately, Material Implication followed by Idempotency would give the conclusion.) Direct derivation would be easier:

1.	$(M \to N) \,\&\, (O \to P)$	
2.	$-N \vee -P$	
3.	$-(M \,\&\, O) \to B$	$\vdash B$
4.	$(-N \to -M) \,\&\, (-P \to -O)$	1, Contra. (twice)
5.	$-M \vee -O$	4,2, Dilemma
6.	$-(M \,\&\, O)$	5, DeM.
7.	B	3,6, M.P.

Nevertheless, indirect proof will always work, even if sometimes inefficiently.

3.11 Tautological Derivation

A tautology, we know, is any sentence which is necessarily true. A tautology can be used as a valid conclusion from any set of sentences. (Please review problem 2 in Exercise 2.4!) Well, then: Suppose some argument has a tautology for a conclusion. For example, $[(A\rightarrow B)\&A] \rightarrow B$ is a tautology, as you may verify with a truth table. Given any set of premises, it will validly imply this (and any other) tautology. That is, for any set of premises (call it *s*), $s \vdash [(A \rightarrow B) \& A] \rightarrow B$ is valid. We may use the method of Conditional Proof to demonstrate exactly that.

1.	s		Any set of premises
2.		(A → B) & A	Cond. Assump.
3.		A → B	2, Separation
4.		A	2, Separation
5.		B	3,4, M.P.
6.	[(A → B) & A] → B		2 → 5, C.P.

Notice that the premises, *s*, were not even necessary for the proof. We could have left them out. Let's do that. We find then that line 1 would now be blank. Let's renumber the steps and dispense with the vertical and horizontal lines which belong to the "main" proof. Here is the result:

1.		(A → B) & A	Cond. Assump.
2.		A → B	1, Separation
3.		A	1, Separation
4.		B	2,3, M.P.
5.	[(A → B) & A] → B		1 → 4, C.P.

The moral of the story is this: To treat the original premises of a derivation (or their conjunction) as a conditional assumption and then to work a conditional proof with them is to produce a tautology.

Here is another example: Given the argument $[(A\vee B)\rightarrow C]\&-C \vdash -A$, we can transform it into a tautology in this way:

1.		[(A ∨ B) → C] & −C	⊢ −A	
2.		(A ∨ B) → C		1, Separation
3.		−C		1, Separation
4.		−(A ∨ B)		2,3, M.T.
5.		−A & −B		4, DeM.
6.		−A		5, Separation
7.	{[(A ∨ B) → C] & −C} → −A			1 → 6, C.P.

This may be done as well in cases of arguments having more than one premise, since those premises may be expressed as a conjunction. In the case of the example above, there could have been these two premises:

1. (A ∨ B) → C
2. −C

To reform a proof into one large conditional proof is simply to assert that a conditional sentence, whose antecedent is the premise (or the conjunction of the premises) of a valid argument, and whose consequent is the conclusion of that argument, must be a tautology. That is, if $s \vdash p$ is valid, then $s\rightarrow p$

is a tautology. What's more, if $s \rightarrow p$ is a tautology, then $s \vdash p$ is valid. When these relationships are exploited using Conditional Proof, we will call the process ***Tautological Derivation***, and we will indicate that such a derivation is possible by writing the tautology with a "⊢" in front of it, but without any premises, thus signifying that the tautology can be derived from any (or no) premises at all.

Example: Prove $\vdash -A \rightarrow (A \rightarrow B)$.
Proof:

1.	$-A$	
2.	$-A \vee B$	1, Weak.
3.	$A \rightarrow B$	2, M.I.
4.	$-A \rightarrow (A \rightarrow B)$	1 → 3, C.P.

In Step 1, the premise is simply the antecedent of the tautology. Now we try to derive the consequent. Once that is done (line 3 in the example), we may treat all the lines as part of a Conditional Proof. The result is line 4, which is the tautology we were trying to derive.

Any tautological derivation yields a tautology. But some tautologies are not expressed as conditional sentences; for example: $-(A \& -A)$. In this case we could use the Equivalence Rules to transform the sentence into $-A \vee A$ and that sentence, in turn, into $A \rightarrow A$. We could provide the derivation for $A \rightarrow A$ and then change that back into $-(A \& -A)$:

1.	A	
2.	A	1, Rep.
3.	$A \rightarrow A$	1 → 2, C.P.
4.	$-A \vee A$	3, M.I.
5.	$-(A \& -A)$	4, DeMorgan's

Exercise 3.11

* Answers to starred problems are given in Appendix D.

Prove each of the following.

* 1. $\vdash (A \& B) \rightarrow (A \vee B)$
2. $\vdash (A \vee B) \rightarrow (-B \rightarrow A)$
3. $\vdash (A \rightarrow B) \rightarrow (-B \rightarrow -A)$
4. $\vdash [(A \rightarrow B) \& (C \rightarrow D)] \rightarrow [(A \vee C) \rightarrow (B \vee D)]$
* 5. $\vdash [(A \rightarrow B) \& (C \rightarrow D)] \rightarrow [(A \& C) \rightarrow (B \& D)]$
6. $\vdash A \rightarrow (B \rightarrow A)$
7. $\vdash -(A \rightarrow B) \rightarrow (B \rightarrow A)$
8. $\vdash (A \& -A) \rightarrow B$
* 9. $\vdash B \rightarrow (-A \rightarrow -A)$
10. $\vdash (C \& D) \rightarrow -(H \& -H)$

3.12 Normal Forms Revisited

The connection between Disjunctive Normal Form and Conjunctive Normal Form may be understood by examining one version of DeMorgan's Laws:

$$p \lor q \equiv -(-p \,\&\, -q)$$

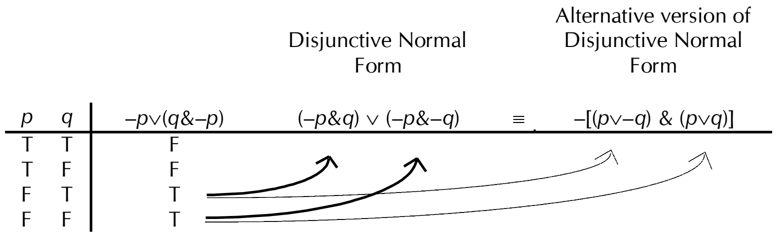
If the left side is true, then the right side must be true as well. But that means that the expression following the "−" on the right side must be the opposite of true, i.e., false.

$$\underbrace{p \lor q}_{\text{true}} \equiv \underbrace{-(\overbrace{-p \,\&\, -q}^{\text{false}})}_{\text{true}}$$

We may state this form of DeMorgan's Law rather awkwardly (but accurately) as: Any disjunction is logically equivalent to the denial of a conjunction of the denials of the original disjuncts. So if the original disjuncts are, say, *p* and *q*, then their denials are −*p* and −*q*, and a conjunction of those denials is −*p*&−*q*, and the denial of that conjunction is therefore the sentence −(−*p*&−*q*).

The left side of the equivalence above represents the general arrangement of Disjunctive Normal Form—i.e., a series of disjunctions—, and the right side represents the *denial* of the general form of Conjunctive Normal Form—i.e., the denial of a series of conjunctions. This works because in constructing Disjunctive Normal Form, we examine those cases where the sentence is true, whereas with Conjunctive Normal Form we examine those cases where the *denial* of the sentence is true (i.e., where the sentence is false). That is, the first denial sign on the right is implicit in the entire construction of Conjunctive Normal Form, because Conjunctive Normal Form is created by attending only to the *false* cases.

You can use DeMorgan's insight to create a Disjunctive Normal Form sentence in a different way: Attend as usual to the cases where the sentence is true, but instead of creating a disjunction of the conjunctive representation of those cases, create a denial of the conjunction of the disjunctive denials of those cases. For example, suppose we want to create the Disjunctive Normal Form for the sentence −*p*∨(*q*&−*p*).



			Disjunctive Normal Form		Alternative version of Disjunctive Normal Form
p	*q*	−*p*∨(*q*&−*p*)	(−*p*&*q*) ∨ (−*p*&−*q*)	≡	−[(*p*∨−*q*) & (*p*∨*q*)]
T	T	F			
T	F	F			
F	T	T			
F	F	T			

The sentence above in Disjunctive Normal Form and the sentence above in the alternative version of Disjunctive Normal Form are logically equivalent, as you can readily verify by converting one into the other by means of DeMorgan's Laws. (The alternative version is not, strictly speaking, in Disjunctive Normal Form; rather, it was created by a procedure which is an alternative to Disjunctive Normal Form, and the sentence can be easily transformed into strict Disjunctive Normal Form by means of DeMorgan's Laws.)

Similarly, you can create a sentence in Conjunctive Normal Form by a similar alternative procedure. Attend as usual to the cases where the sentence is false, but instead of creating a conjunction of the disjunctions of the denials of those cases, create a denial of the disjunction of the conjunctive version of those cases. For example, given the same sentence as above, namely, $-p\vee(q\&-p)$, the Conjunctive Normal Form would be $(p\vee q)\&(p\vee -q)$ (which is the denial of the alternative version of Disjunctive Normal Form), and the alternative construction would yield $-[(-p\&q)\vee(-p\&-q)]$ (which is the denial of the Disjunctive Normal Form), which can easily be converted to strict Conjunctive Normal Form by means of DeMorgan's Laws.

3.13 More Connectives

Alternative Denial

In §1.9 we mentioned that there are sixteen possible, different binary connectives. Since all sixteen can be expressed in terms of denials ("−"), conjunctions ("&"), and disjunctions ("∨") (using either Disjunctive or Conjunctive Normal Form), any other connectives are not really necessary; they are used as a matter of convenience. (In this chapter we have investigated some formal rules for the conversion of expressions from one connective to another.) Is it possible to do better than that? Is it possible to do the work of all sixteen connectives using only one connective (instead of three)? That is, can we find a connective on the basis of which we can define "−", "&" and "∨"? (Then "→", "↔" and so on can be built out of *them*.) The answer is, Yes. One such connective is called ***alternative denial***, or, more popularly, the ***Sheffer Stroke***: "|". Its truth table is column nine from the 16-column truth table in §1.9, and it represents the denial of the "&". (It is therefore, especially in the field of digital electronics, called the ***nand*** function — "nand" from "not" and "and".)

	p	q	p | q
1.	T	T	F
2.	T	F	T
3.	F	T	T
4.	F	F	T

That truth table may be summarized: *The alternative denial is true just in case at least one of the sentences is false.*

You will notice that the truth table for "|" is identical to the truth table for $-(p\&q)$. Look at rows 1 and 4 of the truth table above. If we omit rows 2 and 3, then we would be considering those cases where p and q have the same truth values; if they're both true, then $p|q$ is false, and if they're both false, then $p|q$ is true. Now consider a single sentence, p, by itself and construct the truth table for $p|p$. The definition of "|" says that when both sentences are true, then the "|" produces false, and when both sentences are false, then the "|" produces true. But both sentences in this case are the same, namely, p. Hence:

p	p|p
T	F
F	T

But that truth table is exactly the truth table for $-p$. Hence, to use the "|" to represent the denial of a sentence, simply repeat the sentence with "|" between them. For example,

$-(A \& -B) \quad \equiv \quad (A \& -B) \mid (A \& -B)$
$-[(C \& D) \rightarrow R] \quad \equiv \quad [(C \& D) \rightarrow R] \mid [(C \& D) \rightarrow R]$

and so on.

Since $p|q$ is the NAND function, it means $-(p\&q)$. Hence, an undenied conjunction, $p\&q$ is $--(p\&q)$, or $-(p|q)$. In order to express that sentence using only "|" (that is, without even using "−"), we need to form the denial of $p|q$. Since the denial of a sentence using only "|" is simply a repetition of the sentence with a "|" in between, we have:

$$(p\&q) \equiv -(p|q) \equiv (p|q) \mid (p|q) \, .$$

Now that we can form denials and conjunctions using only the "|", we can use DeMorgan's Laws to create disjunctions. And we can use Material Implication and other equivalence rules to create conditionals, biconditionals, etc.

Joint Denial

Another connective which alone will do the work of all the others is "↓", called ***joint denial***, whose truth table is:

	p	q	p ↓ q
1.	T	T	F
2.	T	F	F
3.	F	T	F
4.	F	F	T

That truth table may be summarized as: *The joint denial is true only when both sentences are false.*

The joint denial represents the denial of the "∨". It is therefore often called the ***nor*** function (from "not" and "or"). In order to transform sentences using "−", "→", etc. into sentences using only "↓", you ought to consider the same sorts of things as we did above with "|".

Exercise 3.12

* Answers to starred problems are given in Appendix D.

Express each of the sentences in 1 through 5 solely in terms of "|".

* 1. A ∨ B
2. −(A ∨ B)
3. −A & B
* 4. A → B
5. A ↔ B

Express each of the sentences in 6 through 10 solely in terms of "↓".

6. −(A & B)
7. −(A ∨ B)
* 8. −(A → B)
9. −(A ↔ B)
10. A & B

Chapter 3 Test

1. Annotate (i.e., give the justification for) each line in the following proof:

    1. $(A \rightarrow B) \,\&\, (C \rightarrow D)$ $\vdash (A \,\&\, C) \rightarrow (B \vee D)$
    2. $A \rightarrow B$
    3. $-A \vee B$
    4. $(-A \vee B) \vee D$
    5. $-A \vee (B \vee D)$
    6. $[-A \vee (B \vee D)] \vee -C$
    7. $-C \vee [-A \vee (B \vee D)]$
    8. $(-C \vee -A) \vee (B \vee D)$
    9. $(-A \vee -C) \vee (B \vee D)$
    10. $-(A \,\&\, C) \vee (B \vee D)$
    11. $(A \,\&\, C) \rightarrow (B \vee D)$

2. Prove: $A \rightarrow B,\ B \rightarrow C,\ M \rightarrow (A \,\&\, -C)\ \vdash\ M \rightarrow S$

3. Prove: $(A \,\&\, C) \rightarrow P,\ -Q \vee (A \,\&\, C),\ (P \vee -Q) \rightarrow S\ \vdash\ (Q \rightarrow S) \vee (L \rightarrow S)$

4. Prove: $\vdash\ \{[M \rightarrow (O \,\&\, P)] \,\&\, (-O \,\&\, M)\} \rightarrow -M$

5. Translate and prove:

    Fred will be convicted if John lies. Here's why: If Sally tells the truth or if George tells the truth, or both, then Fred will be convicted. If John lies, George will go free. George will go free only if either Sally tells the truth or Sam lies, or both. But Sam does not lie.

6. Translate and prove:

    If it is false that both lions and tigers are found in savanna regions, then ocelots are not found there either. In fact, tigers are not found in savanna areas, so ocelots aren't.

7. Translate and prove:

    Galaxy M22 is not very close to our own galaxy. If Reiner's studies are definitive, then Edward's estimates are correct. If no one has complained, then Reiner's studies are definitive. If Edward's estimates are correct and Aldrich's calculations are in error, then Galaxy M22 is very close to our own galaxy. No one has complained. Therefore Aldrich's calculations are not in error.

8. Translate and prove using Indirect Proof:

    Both singles bars and sports pubs are very popular in Portugal—or else the recent report got it all wrong. If singles bars are popular in Portugal, then if sports pubs are popular there, we're going to be rich. So if the recent report did not get it wrong, we're going to be rich.

9. The elementary argument form Modus Tollens might be rendered in rather stilted (but nevertheless accurate) English in this way: "Given any conditional, and given the denial of its consequent, you may validly infer the denial of its antecedent." Render into accurate English (without the use of any of our symbolic notation such as *p*, *q*, "→", etc.) the argument form Hypothetical Syllogism.

10. (a) Construct the Disjunctive Normal Form for $-(A \rightarrow B) \leftrightarrow A$.
    (b) Express the result solely in terms of "|".

— 4 —

The Tree Method

4.1 Consistency and Decomposition

Consider a conjunction: it has two conjuncts. A conjunction can be true only in case both conjuncts can be true at the same time; otherwise the conjunction cannot be true.

	p	q	p & q
1.	T	T	T
2.	T	F	F
3.	F	T	F
4.	F	F	F

To say that *p* and *q* can be true together is to say that they are consistent; two sentences which cannot be true at the same time are inconsistent. The conjunction *p*&*q* is consistent if and only if the two sentences *p* and *q* form a consistent set of sentences.

Consider now a disjunction. It has two disjuncts. And a disjunction can be true only if either or both of its disjuncts can be true; otherwise, the disjunction cannot be true.

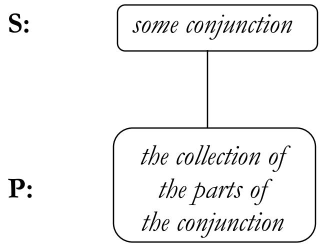
	p	q	p ∨ q
1.	T	T	T
2.	T	F	T
3.	F	T	T
4.	F	F	F

To say that *p* can be true or that *q* can be true (or both) is to say that *p* is consistent or that *q* is consistent (or both), because a sentence which cannot be true is an inconsistent sentence. We might say that a disjunction is consistent if and only if either (or both) of its disjuncts is consistent. If, then, we wish to determine the consistency of $p \vee q$, we could test the consistency of *p*, or, alternatively, test the consistency of *q*. One or the other (or both) will have to be consistent if $p \vee q$ is to be consistent.

In summary:

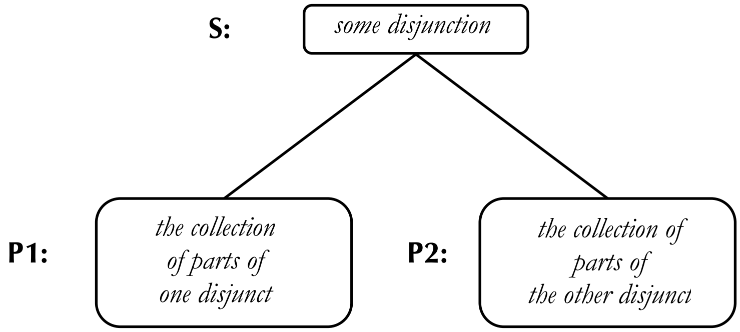
1. If a conjunction, $p\&q$, is consistent, then *p*, *q* is a consistent set of sentences.
2. And *vice-versa*: if *p*, *q* is a consistent set of sentences, then $p\&q$ is a consistent conjunction.
3. If a disjunction, $p \vee q$, is consistent, then either *p* is consistent, or *q* is consistent, or both.
4. And *vice-versa*: if either *p* is consistent or *q* is consistent, or both, then $p \vee q$ is a consistent disjunction.

A truth table is a sure-fire method of determining the consistency of conjunctions and disjunctions. But, as we know, truth tables can become unwieldy beasts. It would be convenient to devise a method for doing what a truth table does, but without the concomitant complexities. As far as consistency of conjunctions and disjunctions goes, we are interested only in those truth table rows which have T in the final column, i.e., those rows indicating consistency. A diagrammatic method for (1) picking out such rows if they exist, and (2) telling us what those rows are (i.e., what truth values for *p* and *q* will yield T in the final column) is available. It is called the ***Method of Truth Trees***, or simply the ***Tree Method***. This is a method for checking the consistency of a sentence by checking the consistency of its parts. That is, the tree method will test the consistency of a conjunction by testing the simultaneous consistency of its conjuncts; and it will test the consistency of a disjunction by separately testing the consistency of its disjuncts. (Hence, the tree method is capable of testing the consistency of any sentence, since any sentence may be expressed using only "&", "∨" and "–". Remember Disjunctive Normal Form?) It does this by first "decomposing" a given sentence into its parts. Consider a conjunction:

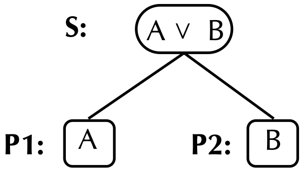


Now, if the sentence in box **S** is consistent, then the set of its elements (in box **P**) is consistent; and conversely, if the set of its elements in box **P** is consistent, then the conjunction in **S** is consistent.

Now consider a disjunction:



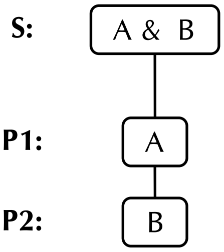
If the sentence in box **S** is consistent, then either the set of the elements in one of its disjuncts (box **P1**) is consistent, or the set of elements in its other disjunct (box **P2**) is consistent; or both. And *vice-versa*. Let's look at the sentence $A \vee B$:



This tells us that if A is true, then $A \vee B$ is true; or if B is true, then $A \vee B$ is true; or, finally, if A is true and B is true, then $A \vee B$ is true. (And this is just what is given in rows 1 through 3 of the truth table for $A \vee B$.)

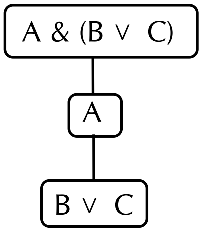
The lines connecting the boxes tell us, in effect, that if every sentence in at least one path below a sentence can be true (i.e., is consistent), then the sentence itself can be true; and if the sentence itself can be true, then all the sentences in at least one path below can be true.

We may "decompose" a conjunction in the following way:



If every sentence down the path can be true, then $A \& B$ can be true; and if $A \& B$ can be true, then so can every sentence down the path.

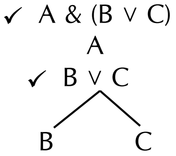
This process of "decomposing" (or analyzing) compound sentences into their constituents may take several steps. For example, here is a conjunction: $A \& (B \vee C)$. It is decomposed by setting one conjunct under the other:



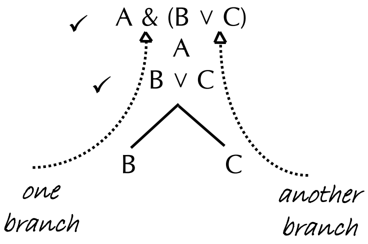
Let us make it a rule that whenever a sentence is decomposed, it will be awarded a check mark (✓). And we might as well get rid of those bothersome boxes, too. In fact, we'll even get rid of the lines joining the sentences, except for the lines which branch from a disjunction. Thus, our ***truth tree*** will now look like this:

✓ A & (B ∨ C)
A
B ∨ C

This truth tree shows us that if $A\&(B\vee C)$ can be true, then both A and $B\vee C$ can be true together; and *vice-versa*. But in order to have more detailed information about the consistency of $B\vee C$, we ought to decompose it, too. The conjuncts of a conjunction are "stacked", one under the other, when a conjunction is decomposed; but the disjuncts of a disjunction are separated into two "branches" when a disjunction is decomposed. Let's do that. (And let's remember to give the disjunction a check mark when we decompose it.) The tree will then look like this:



There are two "branches" to this upside down "tree", but let us call any path from the bottom of a line up to the topmost sentence a "branch". That is, instead of calling that vertical section (which, in the above example, includes $A\&(B\vee C)$, A and $B\vee C$) a "trunk" or something like that, we will say that it forms a part of both branches. Thus, in the above tree, one branch consists of (starting from the bottom) B, $B\vee C$, A, and $A\&(B\vee C)$; and the other branch consists of (starting from the bottom again) C, $B\vee C$, A, and $A\&(B\vee C)$. So our truth trees are upside down, and they have branches but no trunks:



Now, if the elements of at least one branch of the tree below the topmost sentence form a consistent set, then the topmost sentence, $A\&(B\vee C)$, must be consistent (and *vice-versa*). That is, if the left branch, B, $B\vee C$, A is a consistent set, *or* if the right branch, C, $B\vee C$, A is a consistent set, or both, then $A\&(B\vee C)$ is a consistent sentence. Actually, we could have left $B\vee C$ out of consideration, for the consistency conditions for that sentence have been diagrammed by the B branch on the left and the C branch on the right. Thus, if B is consistent, then $B\vee C$ is consistent; so if B, A is a consistent set, then B, $B\vee C$, A is a consistent set. And similarly for the other branch: if C, A is a consistent set, then C, $B\vee C$, A is a consistent set. The check mark (✓), which we give to a compound sentence once it has been decomposed, is in fact a way of saying that the compound sentence is no longer a factor (it is conceptually erased from the tree), because we took account of the conditions under which it is consistent when we decomposed it into its parts.

We can lay this down as a general rule: If all the *atomic* elements *in a given branch* can be true together, then all the compound sentences *in that branch* can be true together as well. This is also shown by the truth table:

	A	B	C	B ∨ C	A & (B ∨ C)
1.	T	T	T	T	T
2.	T	T	F	T	T
3.	T	F	T	T	T
4.	T	F	F	F	F
5.	F	T	T	T	F
6.	F	T	F	T	F
7.	F	F	T	T	F
8.	F	F	F	F	F

In those cases where the simplest elements of the *left* branch (*B*, *A*) of the tree above can be true together (rows 1 and 2 of the truth table), *B*∨*C* is also true, and so is *A*&(*B*∨*C*). In the same way, in those cases where the simplest elements of the *right* branch (*C*, *A*) are true together (rows 1 and 3 of the truth table), *B*∨*C* is also true, and so is *A*&(*B*∨*C*). Therefore, in evaluating a truth tree, we ought to decompose all the sentences into their very simplest elements, and then check to see whether there is at least one branch of the tree in which all the simplest elements which appear in that branch can be true together. If there is such a branch, then we know two things. (1) We know that the original sentence is consistent. And (2) we know an evaluation of the simplest elements which will yield T for that original sentence. That is, in the example tree above, if *B* is true and *A* is true, then *A*&(*B*∨*C*) will be true. We learn that from the left branch. We also know that if *C* is true and *A* is true, then *A*&(*B*∨*C*) will be true. We learn that from the right branch.

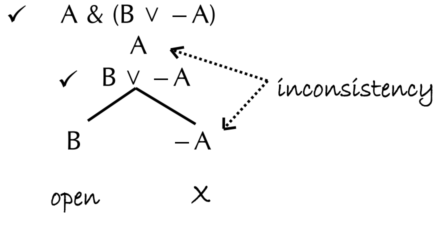
A sentence has been decomposed into its simplest elements when it has been decomposed into single sentence letters (*A*, *B*, *C*, etc.) or their denials (−*A*, −*B*, −*C*, etc.), or both. That is, the sentence *C* cannot be decomposed. Nor can the sentence −*C*.

We have the rule for decomposing a conjunction: Stack one conjunct under the other (and give the conjunction a check mark to show that it has been decomposed). We also have the rule for decomposing a disjunction: Branch one disjunct to the left and the other to the right (and give the disjunction a check mark to show that it has been decomposed). We may introduce another (and rather obvious) decomposition rule, namely, the rule of Double Negation. In other words, erase double dashes. Thus:

✓ − −p
p

This says that if *p* can be true, then −−*p* can be true, and *vice-versa*.

In case a branch contains both a sentence and its own denial, what do we know about the sentences in that branch? We know that they cannot all be true together—they are inconsistent. Here is an example: *A*&(*B*∨−*A*). This is a conjunction, which is decomposed by stacking the conjuncts. Furthermore, one of the conjuncts happens to be a disjunction, which in turn is decomposed by branching. Let's make the resulting tree:



Notice that the right branch contains the atomic sentence *A* (on a line all by itself) and the atomic sentence −*A* (on a line all by itself), and so that branch is not consistent, whereas the left branch is OK. Let us indicate that a branch contains an inconsistency by marking it with "X" and calling that branch ***closed***. So in the example above, the right branch is *closed* and the left branch is *open*. Although there is one closed branch, there is nevertheless an open branch. *The existence of at least one open branch means that there is a way to make the original sentence true, and so the original sentence is consistent.*

Exercise 4.1

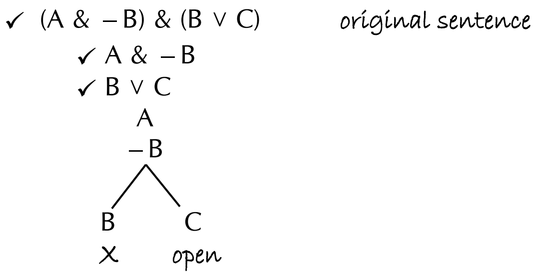
* Answers for starred problems are given in Appendix D.

Make trees for the following sentences.

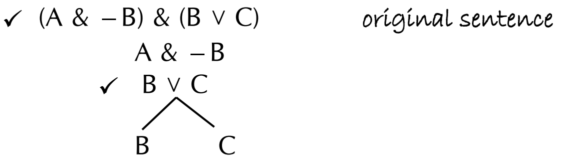
* 1. A ∨ (B & –C)
2. –A & [A ∨ (B & A)]
* 3. [– –C ∨ (A & –C)] ∨ C
4. [(–A & C) ∨ (B ∨ A)] ∨ [–C ∨ (B & A)]
5. (–A & –B) & (B ∨ C)
6. A
7. – – A
8. A ∨ –A
9. A & –A
* 10. (A & –A) ∨ (A ∨ –A)

Let's construct the tree for this sentence (which is similar to the sentence in problem 5 above): (*A*&−*B*)&(*B*∨*C*). It is a conjunction, and so we decompose by stacking the conjuncts. Then we choose either of the resulting sentences and decompose it. Let's choose *A*&−*B*, and then the *B*∨*C* next. Here is the tree:

First version:



But since we are not *required* to choose the *A*&−*B* first, we could just as well have chosen the *B*∨*C* first. Let's do that, just for fun. This is the tree so far:

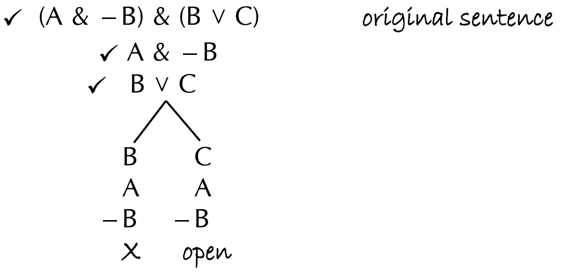


But now a question arises: The *A*&−*B* still needs to be decomposed; but where do we put the results of decomposing it? The answer is: *under each and every open branch below it.* The rule bears repetition:

> ***The results of decomposing a sentence must be placed under each and every open branch below the sentence being decomposed.***

So we must decompose *A*&−*B* and place the resulting stack under the *B* on the left branch; and then we must repeat the procedure for the right branch. In the final tree, we find a *B* and a −*B* in the left branch, which causes that branch to close off. All the atomic sentences in the right branch are consistent with each other, so that branch remains open. Thus:

Second version:



The rule is a most important one, and you must be sure to follow it.

The rule also allows us to neglect the order in which sentences are decomposed. The "first version" tree above tells us that if *C* is true, −*B* is true and *A* is true, then (*A*&−*B*)&(*B*∨*C*) will be true. The other way of doing the tree—the second version above—says the same thing; just read off the atomic elements in any open branch. As long as we follow the rule above, the order in which we decompose decomposable sentences is logically unimportant. The order is, however, sometimes important for strategy: it is usually easier first to decompose conjunctions and then disjunctions. (This is not always so, as we will discover later.)

Finally, the construction of a tree is complete whenever one or both of two things occur: (1) either *all* the branches close (whether or not there still remain decomposable sentences), or else (2) all decomposable sentences have been decomposed.

A word of caution about the first condition. Take the sentence

(−A ∨ B) & (A ∨ B)

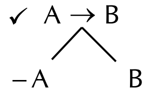
and perform the first step of the decomposition:

✓ (−A ∨ B) & (A ∨ B)
−A ∨ B
A ∨ B

It is not the case that this branch closes (at least, not at this point). It is true that *A* occurs and that −*A* occurs; but in order for the branch to close, *A* has to occur *by itself* (i.e., not as part of a compound sentence), and so does −*A*. The reason is that a closed branch means that there is an inconsistency somewhere in that branch. So, although *A* and $-A$ are inconsistent with each other, they occur in the branch in the tree above only as parts of the compound sentences $-A\vee B$ and $A\vee B$, and these compound sentences are *not* inconsistent with each other. (If you have any doubts, construct a truth table.)

4.2 More Decomposition Rules

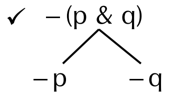
We have decomposition rules for conjunction and disjunction (and the trivial one for double negation). These are all we need to decompose *any* sentence, no matter how complex. (Disjunctive Normal Form, and all that.) Besides, the equivalence rules in section 3.3 give us ways of expressing "→" and "↔" in terms of "&", "∨" and "−". For example, the equivalence rule Material Implication says that $p\rightarrow q$ is equivalent to $-p\vee q$. So if we wish to decompose a conditional such as $A\rightarrow B$, we merely make the tree for $-A\vee B$ instead:

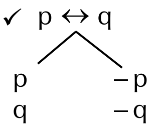


Exercise 4.2

* Answers for starred problems are given in Appendix D.

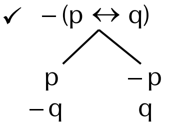
Show that we may use these rules for decomposition:

* 1.


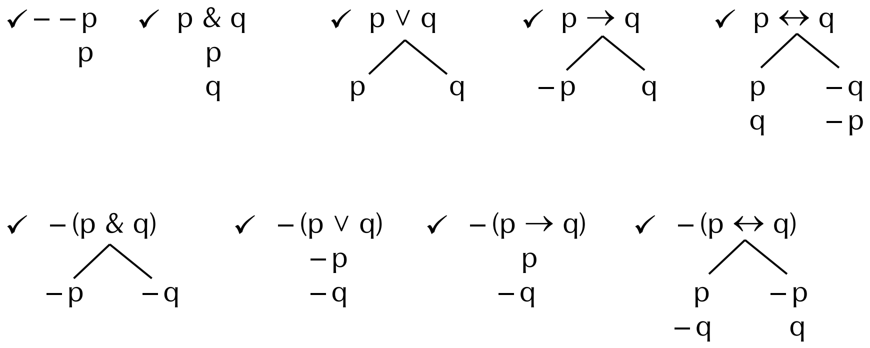
2.


3. ✓ − (p ∨ q)
− p
− q

4. ✓ − (p → q)
p
− q

* 5.


4.3 Summary of Decomposition Rules



Exercise 4.3

* Answers for starred problems are given in Appendix D.

Construct trees for each of the following sentences.

1. D & (D → W)
* 2. −(M → P) ↔ (Q ∨ −M)
3. −[R → (R → R)]
* 4. A & {[(A → B) → B] → −A}
5. S

4.4 Tautologies and Contingent Sentences

The tree method does one thing: it tells us whether a sentence (or set of sentences) is consistent. This one feature of the tree method, however, will allow us to determine other things as well.

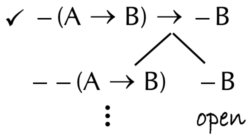
Tautology

If a sentence is a tautology, then it is certainly consistent (that is to say, there will be at least one T in its truth table), and its tree will have at least one open branch. But a contingent sentence also has at least one T in its truth table (and at least one F, to boot), and so its tree will have at least one open branch. So how do we use the tree method to tell the difference between a tautology and a contingent sentence? Simple: The denial of a tautology is inconsistent, whereas the denial of a contingent sentence is consistent. (Remember: A contingent sentence has at least one T and at least one F in the final column of its truth table. So the denial of such a sentence will reverse those values; it will have at least one F and at least one T. But as long as it has at least one T, then by definition it is consistent.) So if you want to use the tree method to determine whether a given sentence is a tautology, just apply the method to the sentence's *denial.* If the denial is inconsistent (the tree closes), then the original sentence is a tautology.

Contingent

If a sentence is contingent, then it is not inconsistent, and the tree method will tell us that. But tautologies are not inconsistent either. So if the tree method tells us that a certain sentence is consistent, we don't know yet whether it is contingent or tautologous. So check to see if the sentence is a tautology or not. That is, apply the tree method to the denial of the sentence as described above. If it turns out not to be a tautology, then you finally know that the sentence must be contingent.

Example: Is $-(A \rightarrow B) \rightarrow -B$ tautologous, contingent, or contradictory? First let's see if the sentence is consistent. If it is consistent, then it will be either tautologous or contingent. If it is not consistent, then, of course, it is contradictory by definition, and there will be nothing more to find out.



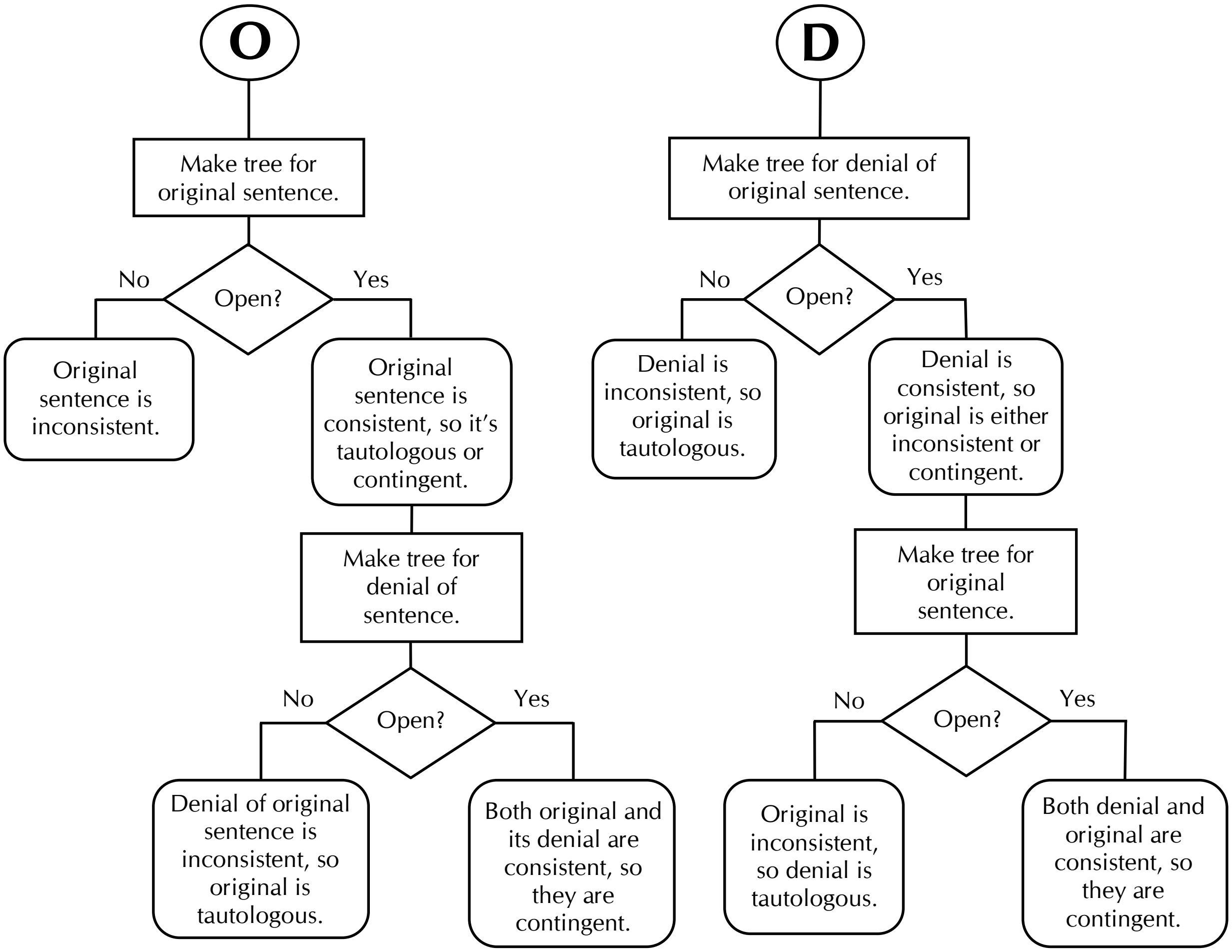
The sentence is consistent. But is it tautologous or contingent? Make the tree for the denial:

✓ $-[-(A \rightarrow B) \rightarrow -B]$
✓ $-(A \rightarrow B)$
✓ $--B$
B
A
$-B$
×

The tree closes, so the denial of the original sentence is inconsistent, and so the original sentence is a tautology. We could have saved ourselves some trouble if we had tested the denial of the sentence first, for then we would have discovered that the sentence was a tautology, and no further tests would have been necessary. On the other hand, if the first tree had closed, we would have discovered the sentence to be inconsistent—a contradiction—and the second tree would not have been necessary. To test the sentence or its denial first, that is the question.

Another example: Is $(A \leftrightarrow -B)\&[-A \rightarrow (A \vee B)]$ tautologous, contingent, or contradictory? If it looked like it might be a tautology, some time might be saved by testing its denial first, and then the original if the denial does not close. But if it looked like it's contingent, or if you had no immediate opinion one way or the other, then you might as well test the sentence first, then the denial if necessary (i.e., if the original does not close). Go ahead and see what happens with this sentence. (You ought to discover that it is contingent.)

The procedure for using the tree method to investigate whether a sentence is tautologous, contingent or contradictory may therefore proceed either by testing the sentence itself (and then its denial, if necessary), or else by testing the sentence's denial (and then the undenied version if necessary). These possibilities may be described by means of the flow chart below. You may begin either at **O** (for testing the original sentence) or at **D** (for testing the sentence's denial).



Exercise 4.4

* Answers to starred problems are given in Appendix D.

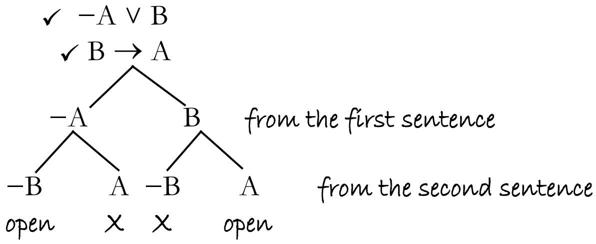
Use the tree method to determine whether the following sentences are tautologous, contingent, or contradictory.

* 1. –[(A & B) ∨ (–A ∨ –B)]
2. [(A ∨ B) & –A] ∨ [(A ∨ B) & A]
* 3. [A & (B ∨ C)] → [(A ∨ B) & (A ∨ C)]
4. A → (–A → –A)
* 5. (A → A) → A
6. [O & (P ∨ S)] & –[(O ∨ P) & (O ∨ S)]
* 7. [(R & –S) ↔ (A → B)] & (B ↔ –R)
8. [(A → –C) & C] → –A
* 9. [A → (–A → B)] ↔ –[B ∨ (B → A)]
10. (A ↔ B) ↔ [–A ↔ (B & C)]

4.5 Testing Sets of Sentences

To test the consistency of a set of sentences, simply list the sentences in a vertical column (well, can there be horizontal columns?), as though they were the result of decomposing a conjunction. (For, after all, if a set of sentences is consistent, then so is their conjunction, and *vice-versa.*) If the tree closes (i.e., if all branches close), then there is no way to assign truth values to the simplest elements to make all the original sentences true at the same time. But if there is *at least one* open branch, then there *is* such a way, and the set is consistent.

For example, you can verify with a truth table that the two sentences $-A \vee B$ and $B \rightarrow A$ are consistent with each other. Here's the tree method proof:



Any open branch will give you a list of the atomic sentences which, if they are all true, would make the original sentence (or set of sentences) true. So the open branch on the left above tells us that if $-B$ is true and $-A$ is true, then the original set of two sentences is true. And the open branch on the right tells us that if A is true and B is true, then the original set is true. (You can get the same information from a truth table—rows 1 and 4—but the tree is easier to construct.)

Exercise 4.5

* Answers to starred problems are given in Appendix D.

Use the tree method to test the consistency of each of the following sets of sentences. If a set is consistent, give a valuation for the atomic elements that will make all the sentences in the set true.

* 1. $-(A \rightarrow B),\ -C \;\&\; -B,\ A,\ A \rightarrow C$
2. $-\{(A \rightarrow B) \rightarrow [A \rightarrow (A \;\&\; B)]\},\ B$
* 3. $A \rightarrow B,\ -(B \rightarrow A)$
4. $-(A \leftrightarrow B),\ -A \vee B,\ -B,\ -(A \rightarrow B)$
* 5. $-(A \leftrightarrow B),\ -A \vee B,\ -B,\ A \rightarrow B$
6. $A \;\&\; B,\ (A \vee C) \rightarrow D,\ -E \vee (A \;\&\; D)$
* 7. $(E \vee F) \rightarrow (G \;\&\; H),\ (G \vee H) \rightarrow I,\ E$
8. $A \vee B,\ -(A \vee B)$
9. $A \vee B,\ A \vee -B$
10. $A \;\&\; B,\ A \;\&\; -B$

The tree method can also be used to determine logical equivalence. Recall that two sentences are logically equivalent if their columns in a truth table are identical. Truth trees can provide the same kind of information. But a caution is in order: if you construct a tree for one sentence and then a second tree for the other sentence, you will have the same problem as you would if you construct one truth table for one sentence and a separate truth table for the other sentence. How will you know whether the trees (or tables) indicate that the sentences are logically equivalent? In some cases that is not difficult to do; if both trees are closed, then you can say that each of the sentences is a contradiction and hence logically equivalent to each other (because all contradictions have the same truth table—nothing but Fs). Or if both trees for the *denial* of the sentences are closed, then you can say that each of the sentences is a tautology and hence equivalent to each other (because all tautologies have the same truth table—nothing but Ts).

But what about contingent sentences? Their trees, and the trees for their denials, are open (which is to say that their truth tables have both Ts and Fs). In such a case, you could examine both trees and convince yourself that the atomic sentences in each open branch of one tree are exactly matched by the atomic sentences in the open branches in the other tree. But doing so for complex trees is both tedious and error-prone. Similarly for using separate truth tables: you'd have to ensure that the order of Ts and Fs in the final column in the one table, as a result of the particular arrangement of Ts and Fs in the rows, is the same as in the other table.

In the case of truth tables, it would be both easier and more reliable to put both sentences on the same table, especially if you can put the final columns for the two sentences next to each other. In the case of the tree method, we can get all the necessary information from a single tree, but we'll have to set it up properly. Here's how: if $p \equiv q$, then $p \leftrightarrow q$ is a tautology, and vice-versa. (That's a nice metalogical claim which you can justify without much trouble.) And if $p \leftrightarrow q$ is a tautology, then the tree for its denial will close.

Another method of using the tree method to test for logical equivalence is available, though in this case two separate trees are constructed. Recall the metalogical claim (see problem 12 in Exercise 2.4) that if $p \equiv q$, then both $p \vdash q$ and $q \vdash p$ are valid (and vice-versa). In the next section we will see how the tree method can determine whether an argument is valid. Once you have seen how that is done, you might be interested in coming back to the discussion here, because it turns out that testing $-(p \leftrightarrow q)$ splits into two branches, one of which is exactly what you would get by testing the validity of $p \vdash q$, and the other of which is exactly what you would get by testing the validity of $q \vdash p$.

Exercise 4.6

* Answers to starred problems are given in Appendix D.

For each of the pairs of sentences below, use the tree method to determine whether the two sentences in the pair are logically equivalent.

* 1. (A → −B), (B → −A)
2. A, A → A
* 3. A, A → −A
4. M → −M, −M → M
* 5. M, M

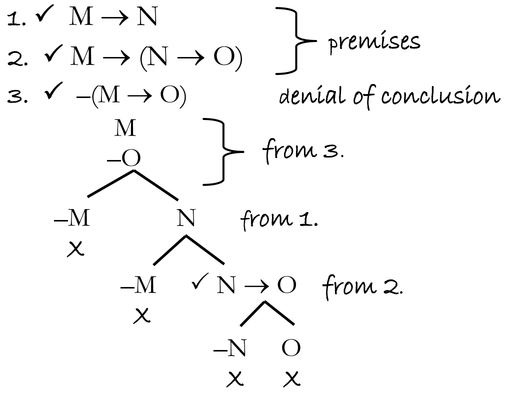
4.6 Validity and Trees

A method for testing consistency will allow us to test the validity of arguments. It will be an application of a kind of indirect proof: to test the validity of an argument, test the set of sentences which form the premises, along with the *denial* of the conclusion. If that turns out to be inconsistent, then the argument must be valid. That is, if p, $-q$ is inconsistent, then $p \vdash q$ is valid. (You may wish to reacquaint yourself with the method of derivation called Indirect Proof. And look again at problem 10 in Exercise 2.4.)

Example: Use the tree method to test the validity of this argument:

$$M \to N,\ M \to (N \to O) \ \vdash\ M \to O$$

If this argument is valid, then the set of sentences consisting of the premises and the denial of the conclusion will be inconsistent, i.e., its tree will close. Here is the completed tree:

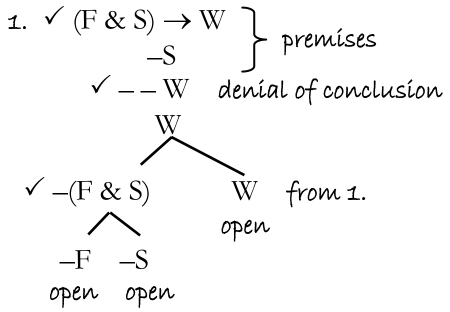


All branches are closed. The argument is therefore valid.

If an argument is invalid, then there must be some valuation of the atomic sentences which will make the premises true and the conclusion false. (Quick refresher: The existence of this possibility defines invalidity; the absence of this possibility defines validity.) A valuation which makes the premises true and the conclusion false will be a valuation which makes all the premises true and the *denial* of the conclusion true as well. The tree method makes use of that metalogical fact; it tests the consistency of the premises with the denial of the conclusion such that if the tree has at least one open branch, the premises with the denial of the conclusion form a consistent set, and therefore the argument is invalid, and any open branch will display a valuation of the kind sought. Such a valuation is called a ***counterexample*** to the argument. An invalid argument will have at least one counterexample. (Obviously, a valid argument can have no counterexample.)

Example: Use the tree method to determine whether the following argument is valid. If it is not, give a counterexample. $(F \,\&\, S) \to W,\ -S \ \vdash\ -W$

The tree:



There is at least one open branch, and so the argument is not valid. Any open branch will provide a counterexample by containing within it a set of atomic sentences which will make the premises true and the conclusion false. Look at the right-most branch, for example. If we just start at the bottom and read off the atomic sentences (ignoring repetitions), then we learn that if *W* is a true sentence and *S* is a false sentence (or *–S* is true), then the argument will have true premises and a false conclusion. The middle branch and the left-most branch are also open, and so they, too, can provide counterexamples. For example, the left-most branch shows us that if *F* is false, *W* is true and *S* is false, then the argument will have true premises and a false conclusion. Any invalid argument will have at least one counterexample.

Exercise 4.7

* Answers to starred problems are given in Appendix D.

Use the tree method to test the validity of the following arguments. For each argument found to be invalid, give a counterexample.

* 1. [(A & O) → O] & [(A ∨ R) → S], (R ∨ B) & A ⊢ O ∨ S
2. (Q ∨ P) → Q, Q → (Q & P) ⊢ (Q ∨ P) → (Q & P)
3. (A → B), (B → C), C → D, –D, A ∨ E ⊢ E
4. –(E & F), (–E & –F) → (G & H), H → G ⊢ G
* 5. (J & K) → (L → M), N → –M, –(K → –N), –(J → –L) ⊢ J
6. C ∨ W, –C & (A → W), –(A → R) ⊢ W ∨ –A
7. A → B, C → D, A ∨ D ⊢ B ∨ C
8. C → K, K → –A, D ∨ A ⊢ –C ∨ D
9. [(D ∨ E) & F] → G, (F → G) → (H → I), H ⊢ D → I
10. (W → –S) & (–W → –Y), X → (–Y → –X), (U ∨ S) & (W ∨ Z) ⊢ X & Z
*11. P → –Q, –Q → R, –R ∨ S, (S ∨ T) → –U ⊢ P → U
12. P → (Q → R) ⊢ (P → Q) → (P → R)
*13. –[(P → Q) & (Q → P)], P ⊢ –Q
14. R ∨ (S & –T), (R ∨ S) → (U ∨ –T) ⊢ T → U
*15. [(P ∨ Q) & (R ∨ S)] → T, –P & –R ⊢ Q → T
16. (P & –Q) → [(R ∨ S) → (T → U)], P & (R ∨ S) ⊢ –Q → (T → U)
17. N → M, M → D, M → P, –P, M ∨ N ⊢ D
*18. P → (Q → R), Q → –S, S ∨ T, –T ∨ U, –U ⊢ P → R
19. P → (Q & R), Q → S, –S ⊢ –P
20. P ↔ Q, –P, Q ∨ R, R → S ⊢ S
21. England will declare war, and either France or Germany will too. If both England and Germany declare war, then neither China nor India will remain neutral. If China will not

remain neutral or if India will not remain neutral, then it is false that both England and France will declare war. Hence, China will remain neutral if and only if India will.

22. If this photo has sentimental value to you, then if you throw it into the trash, you'll regret it. Either you'll regret it or you won't. If you value your past, then the photo has sentimental value to you, but if you want to get rid of unpleasant memories, then you'll throw the photo into the trash. So if you value your past, you will not want to get rid of unpleasant memories.

4.7 Simplifying Complex Expressions

A complex expression may be logically equivalent to a much simpler expression. For example, you can verify with a truth table that the sentence $-p\rightarrow(-q\&-p)$ is logically equivalent to the much simpler expression $p\vee-q$. The desire to simplify a complex expression arises especially in the case of sentences created in Disjunctive (or Conjunctive) Normal Form. Suppose we have a truth table without a sentence to go with it; that is, we want to construct a sentence which will have a certain desired truth table. (For a practical application of this, see Appendix B.) One way to construct such a sentence is by means of Disjunctive Normal Form. For example, let ϕ be some mystery sentence which has this truth table:

p	q	ϕ
T	T	T
T	F	T
F	T	F
F	F	T

If we created the Disjunctive Normal Form expression for ϕ we would get

$$(p \& q) \vee (p \& -q) \vee (-p \& -q)$$

Can that sentence be made simpler? Yes, and by inspecting the truth table you might be able suddenly to see that $p\vee-q$ will do. But inspecting a truth table in hopes of spotting a simpler way of expressing a sentence can be a gamble. Another approach is to use the equivalence rules to manipulate the sentence into some simpler form. Unfortunately, this method, too, relies on noticing which equivalence rules might produce a simpler expression; and sometimes an expression will have to become more complex before it can be simplified.

Nevertheless, a complex sentence can often be fairly easily simplified by means of Commutation, Association, Distribution, DeMorgan's Laws, and two new rules, the first of which we will call E.T. (for "Elimination of Tautology"). The rule E.T. is itself a simplification of a series of other equivalences. Here is how E.T. might be expressed: If one of the conjuncts of a conjunction is a tautology, then it may be eliminated. Consider, for example, the sentence $p\&(q\vee-q)$. One of the conjuncts, $q\vee-q$, is a tautology which will always be true. So the truth value of the entire conjunction will then depend upon the other conjunct, p. That is, if p is true, then the whole conjunct will be true, and if p is false, then the whole conjunct will be false. The $q\vee-q$ contributes nothing to the behavior of the expression which is not already in p, and so we may simply eliminate the $q\vee-q$.

Here is a truth table proof that $q\vee-q$ can be eliminated:

p	q	$q \vee -q$	$p\&(q\vee -q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	T	F

The column under p is identical to the column under $p\&(q\vee -q)$. Hence, the two sentences are identical. You can also provide a derivation of p from $p\&(q\vee -q)$ rather trivially using the rule Separation. And you can provide a derivation of $p\&(q\vee -q)$ from p (try it using Indirect Proof). Those two derivations would constitute still another proof that the two sentences are logically equivalent.

Now consider another sentence which we wish to simplify. Take, for example, the sentence in Disjunctive Normal Form which we created earlier: $(p\&q)\vee(p\&-q)\vee(-p\&-q)$. We may apply a series of equivalence rules (now including E.T.) to it:

1.	$(p\&q) \vee (p \& -q) \vee (-p \& -q)$	
2.	$[(p\&q) \vee (p \& -q)] \vee (-p \& -q)$	1, Association (just to be picky)
3.	$[p \& (q \vee -q)] \vee (-p \& -q)$	2, Distribution
4.	$p \vee (-p \& -q)$	3, E.T.
5.	$(p \vee -p) \& (p \vee -q)$	4, Distribution
6.	$p \vee -q$	5, E.T.

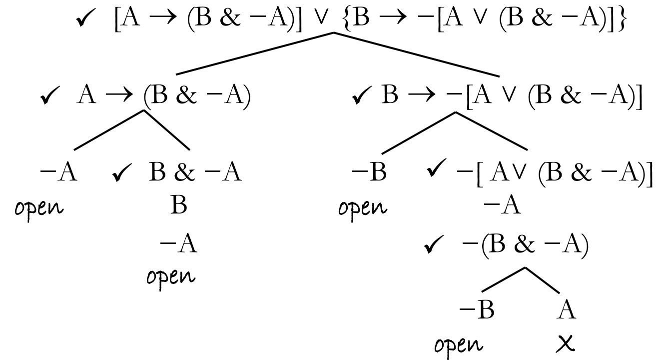
There is another elimination rule which could be helpful. Let us call it "E.C.", for "Elimination of Contradictions". If the earlier rule E.T. allows us to eliminate *tautologies* from *conjunctions*, it is not hard to see how we are justified in eliminating *contradictions* from *disjunctions*. For example, in the sentence $p\vee(q\&-q)$ the $q\&-q$ is always false, and so the entire disjunction is false if p, too, is false, but true if p is true. Hence, the contradiction is redundant; the behavior of the expression depends only on p. (A little investigation ought to convince you that E.T. and E.C. are related by DeMorgan's Laws.) Here is an example of the use of E.C. to help simplify a sentence:

1.	$p \& (q \vee -p)$	
2.	$(p\&q) \vee (p \& -p)$	1, Distribution
3.	$p \& q$	2, E.C.

Two more rules might help occasionally. If one of the conjuncts of a conjunction is a contradiction, then the whole conjunction is necessarily false. In such a case, everything but the contradiction may be eliminated. Let us call this "S.C", for "Simplification of Contradiction". Similarly, if one of the disjuncts of a disjunction is a tautology, then the disjunction as a whole must be a tautology, and so we may eliminate the other disjunct. We can call this "S.T.", for "Simplification of Tautology".

The usual equivalence rules, along with the news rule E.T. (and occasionally E.C., S.C. and S.T.) can be used to simplify complex expressions. The hard part—the part which comes only with experience—is determining which equivalence rule to use at each step. There is always the danger of complicating an expression instead of simplifying it. Some additional assistance can be given by the tree method. Just consider what the tree method basically does: it creates branchings and stackings. The branches are disjunctions, and the stackings are conjunctions. The tree method, then, provides a mechanical method to translate a sentence into disjunctions and conjunctions; it also eliminates contradictions (by closing off branches in which they occur). Once a tree is complete, simply read off all the atomic sentences in an open branch, connecting those sentences together into a conjunction. Then read off all the atomic sentences in the next open branch, again grouping those sentences into their own conjunction. And so on for all the open branches in the tree. Now form the disjunction of all those conjunctions. This final result will be a translation of the original sentence into Disjunctive Normal Form (or something close to it), and, in many cases, it will also be somewhat simplified.

For example, simplify this sentence: $[A\rightarrow(B\&-A)]\vee\{B\rightarrow-[A\vee(B\&-A)]\}$. First construct the tree:



There are four open branches in the tree. Read off the atomic sentences in each branch (ignoring duplicates within a branch), as described above. If you start from the left branch, and proceed bottom-up, then make a disjunction with the next open branch, etc., you get this sentence: $-A\vee(-A\&B)\vee-B\vee(-B\&-A)$. Those four disjuncts may be combined in various ways as needed by means of the equivalence rule Association. The sentence may then be simplified using the equivalence rules, along with E.T., etc. Here is one way to do it, and although the procedure is lengthy, it would have been even lengthier without the initial use of the tree method.

1.	−A ∨ (−A&B) ∨ −B ∨ (−B&−A)	
2.	−A ∨ [(−A&B) ∨ −B] ∨ (−B&−A)	1, Association
3.	−A ∨ [(−B∨−A) & (−B∨B)] ∨ (−B&−A)	2, Comm. and Dist.
4.	−A ∨ (−B∨−A) ∨ (−B&−A)	3. E.T.
5.	(−A∨−A) ∨ −B ∨ (−B&−A)	4, Comm. and Assoc.
6.	−A ∨ −B ∨ (−B&−A)	5, Idempotency
7.	−A ∨ [−B ∨ (−B&−A)]	6, Assoc.
8.	−A ∨ [(−B∨−B) & (−B∨−A)]	7, Dist.
9.	−A ∨ [−B & (−B∨−A)]	8, Idem.
10.	(−A∨−B) & [−A ∨ (−B∨−A)]	9, Dist.
11.	(−A∨−B) & [(−A∨−A) ∨ −B]	10, Comm. and Assoc.
12.	(−A∨−B) & (−A∨−B)	11, Idem.
13.	−A ∨ −B	12, Idem.

Exercise 4.8

* Answers to starred problems are given in Appendix D.

Transform these sentences into simpler sentences containing only "–", "&" and "∨".

* 1. (A → B) & (B ∨ C)
2. (A & B) → (B → A)
3. B & (A ∨ –B)
4. B → (A ∨ –B)
* 5. S → –[(L ∨ D) & (L ∨ –D)]

Construct the Disjunctive Normal Form (or Conjunctive Normal Form) for each of these sentences, and then transform them into simpler sentences containing only "–", "&" and "∨".

* 6. A → (B → –A)
7. –Q ⊕ (S → Q)
* 8. [M ∨ (M → Z)] & –Z
9. (M ↔ S) & (M ∨ –S)
10. –[R → (W → X)] ∨ (W & X)

Chapter 4 Test

1. Construct trees for these sentences:

   a. (C & B) → C
   b. M ↔ –M
   c. –(R ∨ S)
   d. L & (–L ∨ P)
   e. A & (A → A)

2. Use the Tree Method to determine whether these sentences are tautologous, contingent or contradictory.

   a. –(A → C) → [(A ∨ B) → C]
   b. (A → A) → –[(A & B) ∨ (–C & A)]

3. Use the Tree Method to determine whether this set of sentences is consistent.

   –M ↔ F, F → (A ∨ B), A & F, M → (F ∨ A)

4. Use the Tree Method to determine whether these sentences are logically equivalent. (Careful!)

   (A → –C) → (R ∨ F), (–R & –F) → (A & C)

5. Use the Tree Method to test the validity of the following arguments. If an argument is invalid, provide a counterexample.

   a. A → (B & –L), U → (B & –L) ⊢ (A ∨ U) → (B & –L)
   b. (M ∨ S) & –Q, L → (Q & S), L ∨ –S ⊢ –M

6. Translate and use the Tree Method to test for validity. If the argument is invalid, provide a counterexample.

   If either or both of Johnson's or Brown's experiments turn out negative, and if the original data are reliable, then we have wasted millions of dollars on the project. The original data are unreliable only if the samples were contaminated. Johnson's experiment turns out negative, and the samples were not contaminated. Hence, we have wasted millions of dollars on the project.

7. Translate and use the Tree Method to test for validity. If the argument is invalid, provide a counterexample.

   If we commit to the new product line or we restructure our pricing on our existing products, then profits will rise and we can get out of the red. But we do not restructure our pricing on our existing products, because profits do not rise.

8. Is the following two-part claim true? Or is it false? Explain.

   If a sentence is a tautology, then its tree will have all open branches. And *vice-versa*: if a sentence has a tree with all open branches, then it is a tautology.

— 5 —
Quantification

5.1 Subject-Predicate Pairs

Here is an obviously valid argument:

> Aristotle is Greek.
> All Greeks are mortal.
> Therefore, Aristotle is mortal.

The three sentences forming that argument are simple sentences (i.e., not compound), and can be symbolized by single sentence letters in the usual way: $A, G \vdash M$. However, this argument turns out to be invalid when we test it by any of the tools we have so far. (Go ahead; try the tree method, or use a truth table.) Yet surely the argument must be valid.

The problem is not hard to spot. The validity of the argument depends not only on the relationship between the several sentences, but also on the relationship between various *parts* of the sentences. So far, we have no way of symbolizing parts of sentences (unless those parts are themselves complete sentences—which is not the case in the example above).

There are many sentences, such as the three involved in the argument above, which can be analyzed into two main parts: a ***subject term*** and a ***predicate term***. These categories are merely grammatical categories, but they come in handy for logic, too. *A subject term is (or refers to) the main object, thing, entity or being of the sentence.* Thus, in the sentence "Aristotle is Greek", the subject term is "Aristotle". The sentence goes on to say something about Aristotle, namely, that he is Greek. That is, being Greek is given as one of Aristotle's qualities, properties, characteristics or attributes. Being a Greek is *predicated* of Aristotle. In the second premise, "All Greeks are mortal", the subject is "Greeks", and mortality is predicated of them. And in the conclusion, mortality is given as a property of Aristotle.

Let us follow a long-established convention of using upper case letters (*A*, *B*, *C*, etc.) to refer to predicates (qualities, attributes, etc.), and using lower case letters (*a*, *b*, *c*, etc.) to refer to subjects (things, entities, etc.). The convention calls for combining the two symbols together into a pair, with the predicate symbol first and the subject symbol next. Thus, if we let *G* stand for "is Greek" (or "was Greek"; the tenses usually don't concern us) and *a* for "Aristotle", then the sentence "Aristotle is Greek" may be symbolized as *Ga*.

Warning! Predicate letters cannot occur by themselves. If, for example, *G* stands for the predicate "is Greek", then *G* cannot by itself represent a complete sentence (as it used to in the earlier part of this book).

Second Warning! Subject letters cannot occur by themselves. If, for example, *a* stands for the name "Aristotle", then *a* cannot by itself represent a complete sentence.

Moral of These Warnings: Predicate letters must always be paired with subject letters, and *vice-versa*.

These predicate/subject letter pairs can be combined, negated, and otherwise manipulated in the usual manner, because they are, in a word, sentences. For example, to deny that Aristotle is Greek, you need only write: $-Ga$.

Third Warning! Predicate/subject letter pairs can never be split up—nothing is allowed to come between them. On our convention, $G-a$ would be meaningless, because the denial sign can change the truth value only of a sentence, and *a* is a name, not a sentence. $-Ga$ is OK, for we will understand the denial sign to be operating not on *G* (which would also be meaningless), but rather on the whole sentence, just as if we had written $-(Ga)$.

The usual connectives "&", "∨", "→" and "↔" may also be used to connect such sentences. For example, "Aristotle was Greek and Cicero was Roman" could be symbolized as $Ga\&Rc$.

Exercise 5.1

* Answers to starred problems are given in Appendix D.

Use the vocabulary provided to translate the sentences below into English.

Vocabulary:
a : Aristotle   *G* : is Greek
s : Socrates   *R* : is Roman
f : Fido   *D* : is a dog

1. $Ga \; \& \; -Rs$
2. $Da \vee Ds$
3. $Df \rightarrow Ga$

* 4. $Rs \rightarrow (Ra \vee -Df)$

* 5. $Ga \leftrightarrow -Rs$

Using the letters suggested, symbolize these sentences:

6. If ***D***odgson is ***t***ired, then ***A***lice is ***r***eady to leave.
7. Neither ***S***am nor ***G***udrun are ***t***eachers.
8. ***S***ally is both ***t***ired and ***h***ungry.
9. ***K***ing George ***c***ommands and ***J***ohn ***f***alls asleep.

*10. If ***J***ohn ***s***leeps and ***F***lorence ***f***orgets, then either ***L***awrence ***l***eaves or ***M***artha ***m***arries; but (i.e., and) if John does not sleep, then Lawrence marries and Martha forgets.

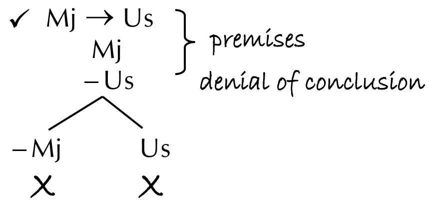
The tree method can be used with this new symbolization. Test the validity of this argument:

> If ***J***oan gets ***m***arried, ***S***am will be ***u***nhappy. Joan gets married. Therefore, Sam will be unhappy.

The argument may be symbolized:

$$Mj \to Us,\ Mj \ \vdash\ Us$$

To test by the tree method, proceed as usual by listing the premises along with the denial of the conclusion.

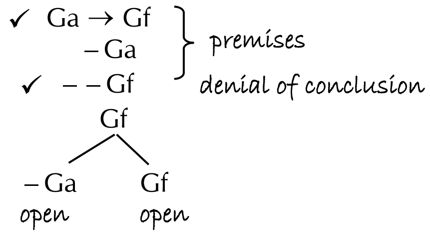


The tree closes, so the argument is valid. (It is, of course, none other than Modus Ponens.)

Another example:

> If ***A***braham is ***G***reek, then ***F***rieda is Greek. Abraham is not Greek. Hence, Frieda isn't.

The tree:



Not all branches close, so the argument is invalid.

Warning! Notice that the left branch in the tree above contains $-Ga$ and Gf. This does not represent an inconsistency, because $-Ga$ is not the denial of Gf. The fact that these two sentences have different subject letters makes them two different sentences. The first one says, "It is false that Abraham is Greek", and the second says, "Frieda is Greek" (or "It is true that Frieda is Greek"). And there is no contradiction in that.

Although we are now able to symbolize parts of simple sentences, we are still unable to prove valid the argument about Aristotle at the beginning of this section. The reason is that there are still other kinds of parts which sentences may have, but which we cannot yet symbolize. We will discuss them in upcoming sections.

Exercise 5.2

* Answers to starred problems are given in Appendix D.

Test for validity by the tree method. For any argument found to be invalid, provide a counterexample.

* 1. [(Aa & Ba) → Ci] & [(Aa & Ri) → Sf], (Ri ∨ Ba) & Aa ⊢ Ci ∨ Sf
2. (Ob ∨ Pb) → Ss, Ss → (Ob & Pb) ⊢ (Ob ∨ Pa) → (Ob & Pb)
3. Ca → Lb, Na → Od, Ca ∨ Na ⊢ Lb ∨ Od
4. Cb ∨ Wa, –Ca & (Ab → Wa), –(Ab → Ra) ⊢ Wa ∨ –Ab
* 5. Cf → Ka, –Ka → Ab, Da ∨ Ab ⊢ –Cf ∨ Da

Use the tree method to determine whether each of these sentences is tautologous, contingent or contradictory.

* 6. (Aa & Rb) ∨ (–Ra ∨ –Rb)
7. [(Gf ∨ Ca) & –Gf] ∨ [(Gf ∨ Ca) & Gf]
8. [(Li & –Ls) ↔ (Ab → Ci)] & (Ci ↔ –Li)
* 9. [Mp & (Lb ∨ Lc)] → [(Mp ∨ Lb) & (Mp ∨ Lc)]
10. [Oa & (Pa ∨ Sa)] & –[(Oa ∨ Pa) & (Oa ∨ Sa)]

5.2 Propositional Functions

We are using lower case letters to refer to individual things. These individual things need not be persons; they can be cities, books, corporations, flower pots, scraps of paper, or anything else which can be singled out and named. (Although it may seem strange to give a name such "Fred" to a particular scrap of paper, that is probably only because scraps of paper are usually less important and more transitory than people, whom we routinely give names. But as long as we do give names to people, pets, books, corporations, boats and cities, there is nothing prohibiting us from giving names to anything we wish, including flower pots and scraps of paper.) In any given context, a lower case letter stands for the same particular individual whenever it occurs. Thus, the sentence *Ha*→*Ma* asserts that if a particular thing, *a* (i.e., a particular thing named *a*), has the attribute *H*, then that thing (that same thing) has the attribute *M*. For example: "If Aristotle is human, then Aristotle is mortal." Any particular individual thing can have any number of different attributes.

In addition, a given attribute can apply to any number of different things. *H*, for example, might be predicated not only of *a*, but also of *b*, *c*, *d* and so on. That is, *Ha* might assert that Aristotle is human, *Hb* that Bolzano is human, *Hc* that Cicero is human, and so on. All these sentences have a common pattern:

H__

where the blank is filled in with various lower case letters. Each lower case letter, remember, is the abbreviation of the name of something. (We may call such letters ***individual constants***, or just ***names***.) But let us separate the letters *x*, *y* and *z* (and sometimes *u*, *v* and *w*) from the rest of the alphabet and use them not to fill in the blank, but rather to *represent* a blank; they will be *place holders* for names (constants). That is, instead of writing *H*__, where the blank is merely a placeholder for writing in individual names (*a*, *b*, etc.), we will write *Hx* or *Hy* or *Hz*. We may call *x*, *y* and *z* ***individual variables***.

Of course, *H*__ is not a sentence. It cannot have a truth value until we specify (i.e., give a name to) some individual thing which is supposed to have the attribute *H*, and then the result is true if the thing does have the attribute, and false if the thing does not have the attribute. But if *H*__ is not a bona fide sentence, then neither are *Hx*, *Hy* nor *Hz*, for we are adopting the convention that *x*, *y* and *z* are merely versions of blanks—placeholders—and so they do not name anything. Expressions such as *Hx* are not yet sentences; rather, they are called ***propositional functions*** (or ***sentence functions***), and they become propositions (sentences) when the blank is filled in with a name (constant), i.e., when variable is replaced by a constant. *Hx* is a generalized formulation which has many *instances*: *Ha*, *Hb*, *Hc*, etc. The action of substituting a name for the variable is called ***instantiation***. Thus, a propositional function, which has no truth value—is not a bona fide sentence—can be made into a sentence by instantiation.

5.3 Quantification

In addition to instantiation, there is another way to turn propositional functions into sentences, and that is called ***generalization***—also called ***quantification***. The sentences “Everything has mass” and “Something is baking” do not name any *particular* thing, although they do contain predicates (attributes), namely, “has mass” in the first sentence and “is baking” in the second. Such sentences are called ***general propositions***. Another way of expressing the first sentence is:

> Given any individual thing whatsoever, it has mass.

Or,

> Given any *x*, *x* has mass.

The *x* here is the variable which is the placeholder for constants (names). The last part of the sentence (“*x* has mass”) can be expressed as a propositional function: *Mx*. Thus, we may write:

> Given any *x*, *Mx*.

The phrase “Given any *x*” is customarily symbolized as (∀*x*), and is called the ***universal quantifier***. (Sometimes the universal quantifier is written more simply as (*x*).) Now we may write:

> (∀x)M*x*

The phrase “Given any *x*” can be expressed in many logically equivalent ways. These all mean the same thing:

> Given any *x*, *Mx*.
> All things fall into the class called *M*.
> All things may be categorized as *M*.
> Every *x* has the property *M*.
> All *x* is *M*.
> Everything is *M*.

> For all *x*, *Mx*.
> Every *x* is *M*.
> Each *x* is *M*.

And each of them (all of them, every one of them) may be symbolized as: $(\forall x)Mx$.

There was another sentence I mentioned at the beginning of this section: "Something is baking". "Something", of course, does not mean everything, and so this sentence can not be symbolized as, say, $(\forall x)Bx$, for that would claim that everything was baking. We shall take the word "something" (or the phrase "some things") to mean "at least one thing":

> There is at least one thing which is baking.

Or,

> There is at least one *x*, such that *x* is baking.

Or,

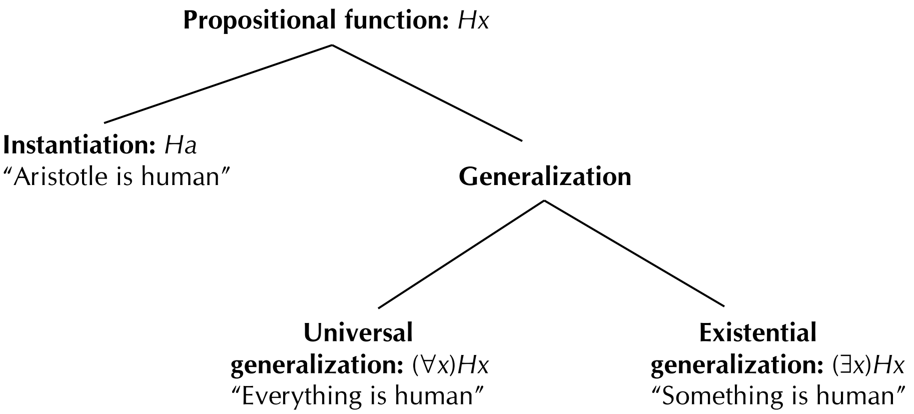
> There is at least one *x*, such that *Bx*.

The phrase "There is at least one *x*, such that..." is conventionally symbolized as $(\exists x)$ and is called the ***existential quantifier***. (Since $(\forall x)$ is called the universal quantifier, perhaps $(\exists x)$ ought to be called the "particular quantifier". But we shall use the term "existential quantifier", for reasons which will be given later.) Now we may symbolize the entire sentence as $(\exists x)Bx$, and it may be translated into English in a variety of logically equivalent ways.

> There is at least one *x* such that *Bx*.
> Some things are *B*.
> Something is *B*.
> Some *x* is (or are) *B*.
> There are *B*s.
> *B*s exist.
> Some *x* can be classed as *B*.

Notice, however, that these do not quite capture the ordinary meaning of "some". To say that some *x* is *B* is sometimes taken to imply (although somewhat weakly) that some *x* is not *B*, too. Yet that implication is not allowed by our symbolization. We will insist that $(\exists x)Bx$ commits us only to the existence of at least one thing that is *B*; and whether there is more than one thing that is *B*—or even whether all things are *B*—is something that is neither affirmed nor denied by $(\exists x)Bx$.

We have two ways of turning a propositional function (such as *Mx*) into a bona fide sentence: by instantiation, i.e., by replacing the variable with some constant name (such as *a*); or by generalization. And we have just discussed two ways of generalizing: universal (e.g., $(\forall x)Mx$) and existential (e.g., $(\exists x)Bx$).



Warning! A variable (*x*, *y*, etc.) can never appear in a sentence unless it has its own quantifier (either universal or existential), and a quantifier can never appear in a sentence unless it quantifies a matching variable. Is this a sentence? $(\exists x)Ha$. No. It is gibberish. There is a quantifier which quantifies no variable. The *a* is a constant—the name of a particular thing—and not a variable. Note, however, that it is not the existence of *a* which makes the sentence gibberish; rather, it is the absence of a variable for the quantifier. As long as a sentence has a quantifier for any variables, and one or more variables for any quantifier, then the sentence is not gibberish. (Whether the sentence also contains individual names such as *a*, *b*, etc., is, as we will see, irrelevant.)

Honest-to-goodnessly quantified sentences are sentences like any other, and they may be treated in the traditional manner: they can be denied and compounded with other sentences by the usual connectives "&", "∨", "→" and "↔".

The expressions which follow a quantifier need not be simple things such as *Mx*. They can also be very complex propositional functions such as $(Bx \to -Ax) \,\&\, (Ux \vee Fx)$, and may also contain sentences with names (constants). But it must be remembered that propositional functions are not sentences, and an expression with a variable in it but no quantifier is, at best, only a propositional function. Notice that in the expression

$$(Bx \to -Ax) \,\&\, (Ux \vee Fx)$$

the propositional functions are connected with "→", "&" and "∨", and that one propositional function even has a denial sign. But how can that be, if propositional functions are not sentences? OK, so I'm making an exception to the rule. Actually, we will develop a very reasonable way of interpreting propositional functions connected by what we used to call sentence connectives.

Although propositional functions can be turned into legitimate sentences by the use of quantifiers, there is a rule about doing that:

> *If a quantifier is followed by an opening parenthesis (or bracket or brace), then the scope (the range of the powers) of that quantifier extends to the closing mate of that parenthesis (or bracket, etc.). If the quantifier is not followed by an opening parenthesis (or bracket, etc.), then its scope extends to, but does not include, the next connective.*

Here are some examples:

1. $(\forall x)[\underbrace{Ax \to (Bx \,\&\, Cx)]}_{\text{scope of } (\forall x)}$

2. $$(\exists x)\underbrace{Ax}_{\text{scope of }(\exists x)} \rightarrow Ba$$

3. $$(\forall x)\underbrace{[Hx \rightarrow (Ax \;\&\; -Bx)]}_{\text{scope of }(\forall x)} \rightarrow (\exists x)\underbrace{(Hx \;\&\; -Bx)}_{\text{scope of }(\exists x)}$$

You can see that the scope of a quantifier is something like the scope of the denial sign.

There is an additional rule:

> *Every variable must fall within the scope of a matching quantifier, and a quantifier governs or refers to only that variable that (1) is coupled with the quantifier and (2) lies within the quantifier's scope.*

Consider this sentence: $(\forall x)(Ax \rightarrow Ay)$. Although the expression Ay falls within the scope of a universal quantifier, it is nevertheless not "governed" by that quantifier, for the quantifier refers to x, whereas the variable is a y. (Even though x, y and so on are variables—placeholders—there can be differences between them, as we shall see later on, just as there can be a difference between variables in algebra.) Thus, the sentence above is only gibberish, because it contains a variable, y, which does not fall within the scope of a matching quantifier.

Quantified sentences may occur within other quantified sentences, as in the following example. (These kinds of complexities may give us trouble translating into English; we'll tackle that issue later.)

$$(\forall x)\underbrace{\{Ax \rightarrow [Hx \;\&\; (\exists y)\overbrace{Gy}^{\text{scope of }(\exists y)}]\}}_{\text{scope of }(\forall x)}$$

In such cases it is nice to have two different variables so that things do not get confusing, although, technically, the sentence $(\forall x)\{Ax \rightarrow [Hx \;\&\; (\exists x)Gx]\}$ is legitimate. By our rules, the Gx would be governed by the $(\exists x)$.

Exercise 5.3

* Answers to starred problems are given in Appendix D.

Determine whether each of the following is a sentence or just gibberish.

* 1. $(\forall x)Hx \rightarrow Gx$
2. $(\exists x)Ba \rightarrow Fx$
* 3. $(\forall x)[(Ha \;\&\; Wa) \rightarrow Sx]$
4. $(\exists x)[(Hx \;\&\; Ma) \;\&\; La]$
5. $Ga \rightarrow (\forall x)Hx$
6. $Ga \rightarrow (\exists x)$
7. $(\forall x)[(Bx \;\&\; Mc) \rightarrow Hx] \rightarrow (Lx \;\&\; -Hx)$
8. $(\forall x)Rx \rightarrow (\forall x)Rx$
* 9. $(\forall x)Rx \rightarrow (\exists y)Ry$
10. $(\forall x)(Hy \rightarrow Sx)$

5.4 Quantifier Duality Rules

"Something is omniscient" may be symbolized as $(\exists x)Ox$. The denial of this claim would be, "It is false that something is omniscient": $-(\exists x)Ox$. But to say that it is false that there is something which is omniscient is to say that nothing is omniscient. Thus, "Something is omniscient" and "Nothing is omniscient" must be contradictory sentences: if one of them is true, the other must be false.

But there is another way of saying that nothing is omniscient: "Given any x, x is not omniscient". Or, "Given any x, $-Ox$". I said before that Ox is a propositional function and, hence, has no truth value. In that case, what is the denial sign denying? We will allow ourselves to use the negations of propositional functions as parts of bona fide sentences as long as the usual rules are followed, namely, the variable must be quantified. Thus, $(\forall x)-Ox$ will be a legitimate sentence which means: "Given any x, it is not omniscient", which, according to the above considerations, is the same as "Nothing is omniscient". Since we've already noted that the sentence "It is false that something is omniscient" is equivalent to the sentence "Nothing is omniscient", then their symbolic versions ought to be equivalent as well. So:

(1) $-(\exists x)Ox \equiv (\forall x)-Ox$

We know that if two sentences are logically equivalent, then so are their denials. So deny the two sentences in the above equivalence pair and you get another equivalence pair:

$$--(\exists x)Ox \equiv -(\forall x)-Ox$$

By Double Negation on the left sentence we get:

(2) $(\exists x)Ox \equiv -(\forall x)-Ox$

We can make a general claim about what's going on here. Instead of talking about a particular propositional function, such Ox, let us generalize and use Φx (Greek letter *Phi*) to indicate any propositional function in which the variable x occurs, such as Ox, $Hx\&Rx$, $Bx\rightarrow Lx$, etc. (By the way, there is nothing magical about Φx; we use it merely because right now we don't care about the inner details of the propositional function. In a similar way, we used lower case letters p, q and so on earlier in this book to represent indifferently any sentence.) Now equivalences **(1)** and **(2)** above may be generalized:

(1′) $-(\exists x)\Phi x \equiv (\forall x)-\Phi x$
(2′) $(\exists x)\Phi x \equiv -(\forall x)-\Phi x$

A similar procedure applies to this sentence: "Something is not omniscient", which may be symbolized as $(\exists x)-Ox$ and is the equivalent of denying that everything is omniscient:

It is false that (given any x, it is omniscient)
−(given any x, it is omniscient)
−(given any x, Ox)
$-(\forall x)Ox$

So now we have this equivalence:

(3) $(\exists x) - Ox \;\equiv\; -(\forall x)Ox$

Once again, if two sentences are logically equivalent then so are their denials. Hence, the denials of the sentences in **(3)** must be equivalent:

$$-(\exists x) - Ox \;\equiv\; --(\forall x)Ox$$

And by Double Negation on the right sentence we get:

(4) $-(\exists x) - Ox \;\equiv\; (\forall x)Ox$

Now, once again let's talk about any propositional function, and not just about *Ox* in particular.

(3′) $(\exists x) - \Phi x \;\equiv\; -(\forall x)\Phi x$
(4′) $-(\exists x) - \Phi x \;\equiv\; (\forall x)\Phi x$

Equivalences **(1′)**, **(2′)**, **(3′)** and **(4′)** are called the ***quantifier duality rules***. These rules allow us to substitute one quantifier for another (with appropriate use of the denial sign). Why would we want to do such a thing? Sometimes we will do it for the sake of economy of expression; and sometimes we will do it just to be perverse; but usually we will do it when using the tree method for quantified sentences, one of the rules of which states that a quantified sentence cannot be decomposed until the quantifier is on the left-most side of the sentence (i.e., not even preceded by a denial sign). But we'll get to that later.

Simple quantified sentences can be represented diagrammatically using Venn diagrams, which can sometimes make the whole business of predication and quantification a bit more understandable. Such diagrams can also be used to test the validity of some arguments. See Appendix A for more details.

Exercise 5.4

* Answers to starred problems are given in Appendix D.

Four of the following expressions have attempted to disguise themselves as sentences. But we know better, don't we? Transform each of the six bona fide sentences into logically equivalent sentences by the use of the quantifier duality rules.

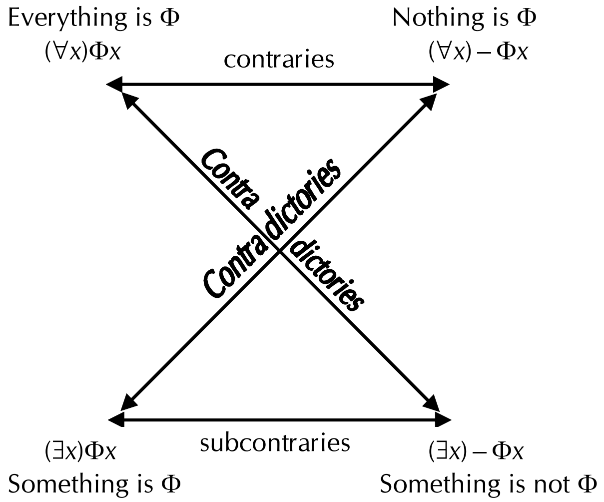
* 1. $(\forall x) - (Hx \rightarrow Ax)$
2. $(\exists x)Rx \rightarrow Lx$
* 3. $-(\forall x)Mx$
4. $(\exists x) - -(Lx \;\&\; -Bx)$
* 5. $(\forall x)La$
6. $-(Mx)(\exists x)$
7. $-(\exists x) - [(Gx \rightarrow Dx) \;\&\; -Ox]$
8. $-(\exists x)Nx$
* 9. $-(\forall x) - (Ax \rightarrow Bx)$
10. $Sx \rightarrow (\forall x)(Ax \rightarrow Bx)$

5.5 Contrariety

We have discussed four different sentences:

Everything is omniscient: $(\forall x)Ox$
Nothing is omniscient: $(\forall x) - Ox$
Something is omniscient: $(\exists x)Ox$
Something is not omniscient: $(\exists x) - Ox$

And we have mentioned the general version of each, using Φx instead of specifically Ox. The relations and interconnections between these types of sentences may be illustrated by this array:



Notice that contradictories are placed at opposite corners. The two top sentences (both universally quantified) are sometimes called ***contraries***. They have this feature: they cannot both be true at the same time, although they can both be false at the same time. As an example, consider the two sentences "Everything is a Buddhist" and "Nothing is a Buddhist". There is no way both of these sentences could be true together. If everything really was Buddhist, then surely it would be false that nothing was. And if nothing was Buddhist, then surely it would be false that everything was. So there is no way that both of these sentences could be true together. But there is a way that both of them could be false. In the world we live in, both of the sentences are false: It is false that everything is a Buddhist, and it is also false that nothing is a Buddhist. Notice that the two sentences are not *contradictories*; if they were, then they would always have *opposite* truth values, which would mean that they could not *both* be true, and they could not *both* be false.

The two sentences at the bottom of the square (both existentially quantified) are called ***subcontraries***, and they have this feature: they cannot both be false at the same time, although they can both be true, as you can understand by once again using the Buddhists as examples. And once again, they are not contradictories, because they do not always have opposite truth values. (By the way, all this presupposes that there is something in the universe. If the universe were really empty—if there were nothing—then these features of contraries and subcontraries would no longer hold. One wonders if anything would—since there would be no anything.)

5.6 Sentences Involving Two Terms

Let's focus attention now on one very common type of quantified sentence. This type has one variable but two predicate letters (say, *A* and *B*). It has four variations:

1. Everything which is *A* is also *B*.
2. Nothing which is *A* is *B*.
3. Something which is *A* is *B*.
4. Something which is *A* is not *B*.

Let's start with the third variation. It says that there is something which is *A* and which is also *B*. Symbolically: $(\exists x)(Ax \mathbin{\&} Bx)$. Here are some examples:

> Some ***a***lligators are ***b***old-faced liars.
> There is at least one thing such that it is an ***a***lligator and a ***b***old-faced liar.
> There are ***a***lligators that are ***b***old-faced liars.
> Some things are ***a***lligators and ***b***old-faced liars.
> Some ***a***dulterers are ***b***usinesswomen who don't take "no" for an answer unless they are confronted with the facts of legal precedent involving extra-marital affairs.

(The purpose of that last, rather lengthy example is only to show that some sentences having complex predicates may nevertheless have simple logical structures.)

Now look at the fourth variation: $(\exists x)(Ax \mathbin{\&} -Bx)$. Examples:

> There are ***a***dulterers who are not ***b***usinesswomen.
> Some ***a***dults are not ***b***usybodies.
> At least one ***a***lligator is not a ***b***old-faced liar.

Next, look at the first variation: "All *A* is *B*". We can interpret such claims in this way: "Given any *x*, if it is *A*, then it is *B*". Symbolically: $(\forall x)(Ax \rightarrow Bx)$. Why do we use conditionals? For two reasons. (1) Suppose I claim "All avocados are brown". Now, clearly I don't mean to say that everything in the universe is an avocado (and brown), so my claim should not be symbolized as $(\forall x)(Ax \mathbin{\&} Bx)$. Rather, I'm saying something like: "*If* you find something which is an avocado, *then* you will have found something which is brown." (2) The second reason is this: How could you disprove my claim that all avocados are brown? Simple. Just find an avocado which is not brown. You could then make the claim:

$$(\exists x)(Ax \mathbin{\&} -Bx)$$

Well, if your claim is the denial of my claim, then this equivalence must hold:

$$\underbrace{(\exists x)(Ax \mathbin{\&} -Bx)}_{\text{Your claim}} \qquad \equiv \qquad \underbrace{-(\forall x)(Ax \rightarrow Bx)}_{\text{Denial of my claim}}$$

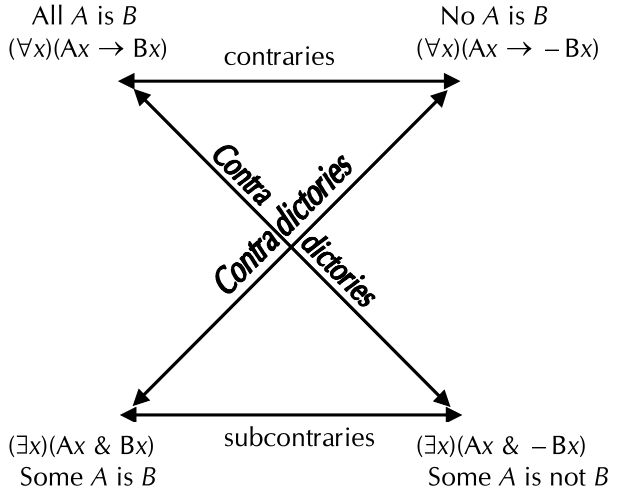
We can satisfy ourselves that these are indeed equivalences by using the Equivalence rules along with the quantifier duality rules. Let's start with the denial of my claim. We should be able to transform it into your claim, if the two really are logically equivalent. And here's a way to do it.

$-(\forall x)(Ax \to Bx)$	denial of my claim
$(\exists x)-(Ax \to Bx)$	duality rule
$(\exists x)-(-Ax \vee Bx)$	Material Implication
$(\exists x)(Ax \;\&\; -Bx)$	DeMorgan's

Of course, the procedure works in reverse, too. In addition, the denial of your claim can be transformed into my claim (and vice versa). Hence, our claims are the opposites of each other.

Finally, let's look at the second variation: "Nothing which is *A* is *B*". We could just say, "All *A* is not *B*", which could be symbolized as $(\forall x)(Ax \to -Bx)$. Or, we could note that "No *A* is *B*" is the contradictory of "Some *A* is *B*". So if we deny "Some *A* is *B*" we ought to be asserting something logically equivalent to "No *A* is *B*". Try transforming one into the other following a procedure similar to that used above.

These four variations above fit onto that array shown before:



Exercise 5.5

* Answers to starred problems are given in Appendix D.

Symbolize these sentences, using *E* for "is an electron" and *P* for "is a particle".

1. All electrons are particles.
2. Every electron is a particle.
* 3. Each electron is a particle.
4. If something is an electron, then it is a particle.
* 5. Nothing is an electron unless it is a particle.
6. Only electrons are particles.
7. Not all electrons are particles.
8. Some electrons are not particles.
9. Some electrons are particles.
10. No electrons are particles.
11. No electron is a particle.
* 12. There are no electrons which are particles.
13. There are no electrons.
14. There is at least one electron which is a particle.
15. There is at least one particle which is not an electron.
16. There is no such thing as an electron which is not a particle.

17. If all electrons are particles, then all particles are electrons.
18. All electrons are particles, but not all particles are electrons.
* 19. Something is an electron only if it is a particle.
20. If some electrons are particles, then no electrons are non-particles.

5.7 Some Hints About Translation

Sentences can be made from propositional functions either by instantiation or by quantification (universal or existential). Clearly, then, there can be sentences which involve both. For example, the sentence "If ***E***instein is ***c***orrect, then all ***q***uantum physicists are on the ***w***rong track" is a conditional sentence and ought to be symbolized as one:

(Einstein is correct) → (all quantum physicists are on the wrong track)
$\mathrm{Ce} \rightarrow$ (all quantum physicists are on the wrong track)
$\mathrm{Ce} \rightarrow (\forall x)(\mathrm{Q}x \rightarrow \mathrm{W}x)$

Another example: "If ***E***instein is ***c***orrect, and if some ***t***heories are ***r***estricted in scope, then no ***p***hysicist is ***f***ully satisfied". This, too, is a conditional sentence:

[(Einstein is correct) & (some theories are restricted in scope)] → (no physicist is fully satisfied)
[Ce & (some theories are restricted in scope)] → (no physicist is fully satisfied)
$[\mathrm{Ce} \,\&\, (\exists x)(\mathrm{T}x \,\&\, \mathrm{R}x)] \rightarrow$ (no physicist is fully satisfied)
$[\mathrm{Ce} \,\&\, (\exists x)(\mathrm{T}x \,\&\, \mathrm{R}x)] \rightarrow (\forall x)(\mathrm{P}x \rightarrow -\mathrm{F}x)$

Sometimes it is tempting to include two sentences within the scope of the same quantifier, when actually they should be separated. For example, "Some dogs are brown and some aren't" might mistakenly be symbolized as

$$(\exists x)[(\mathrm{D}x \,\&\, \mathrm{B}x) \,\&\, (\mathrm{D}x \,\&\, -\mathrm{B}x)]$$

This is a mistake because each occurrence of x must refer to the same thing within the scope of a given quantifier, and so the x in $Dx\&Bx$ must refer to the same thing as the x in $Dx\&-Bx$. If we translated the symbolic form above faithfully back into English we would get: "There is at least one thing such that it is a dog and brown and a dog and not brown". Clearly a contradiction. What we really need is two sentences, not one:

(Some dogs are brown) & (some dogs are not brown)
$(\exists x)(\mathrm{D}x \,\&\, \mathrm{B}x)$ & (some dogs are not brown)
$(\exists x)(\mathrm{D}x \,\&\, \mathrm{B}x) \,\&\, (\exists x)(\mathrm{D}x \,\&\, -\mathrm{B}x)$

More translation hints will be given later.

Exercise 5.6

* Answers to starred problems are given in Appendix D.

Translate each of the following sentences, using the letters indicated.

1. ***S***mith is ***i***ll and no ***d***octor is ***a***vailable.
2. If some ***h***elp is ***a***vailable, then ***S***mith will not ***d***ie.
* 3. If all ***l***awyers are ***r***ich, and no ***d***octor is ***t***rustworthy, then ***F***arnsworth is rich and trustworthy.
* 4. ***G***reen is a ***p***sychiatrist, and if only ***h***onest people are ***t***rustworthy and ***n***oble, then Green is neither trustworthy nor noble.
* 5. If ***l***awyers are ***w***ell-off only if they have ***c***lients, and if ***A***aron is both a lawyer and well-off, then Aaron has clients.
6. If ***A***ristotle was a ***l***ogician, then some ***p***hilosophers are logicians.
7. If ***A***ristotle was not a ***p***hilosopher, then no ***l***ogicians are philosophers.
8. If both ***A***ristotle and ***S***ocrates were ***p***hilosophers, then some philosophers are ***l***ogicians and some are not.
* 9. If only ***m***athematicians are ***p***hilosophers, and if ***R***ussell was a philosopher, then Russell was a mathematician.
10. If ***p***hilosophers are ***m***athematicians only if they are ***l***ogicians, then some philosophers are neither mathematicians nor logicians.

Translate each of the following, using this vocabulary: M*x*: *x* is a mammal. H*x*: *x* is a horse. O*x*: *x* is an ostrich. L*x*: *x* is larger than an elephant. f: Fred. o: Olga.

11. $-(\forall x)(Mx \rightarrow Hx)$
12. $(\exists x)(Mx \ \& \ -Hx)$
13. $-(\exists x)(Mx \ \& \ Hx)$
14. $Hf \ \& \ [(\forall y)(Hx \rightarrow Mx) \rightarrow Mf]$
15. $-(\exists x)(Ox \ \& \ Hx)$
*16. $(\forall x)(Ox \rightarrow -Lx)$
*17. $-(\forall x)(Ox \rightarrow Lx)$
18. $-Of \ \& \ Oo$
*19. $Ho \leftrightarrow (\forall x)(Ox \rightarrow Hx)$
20. $(\exists x)(Hx \ \& \ Lx) \rightarrow (\exists x)(Ox \ \& \ Lx)$

Chapter 5 Test

1. Which of the following are bona fide sentences, and which are gibberish?
   a. $Bc \rightarrow (Me \& Fe)$
   b. $(\forall x)Ux \rightarrow Ra$
   c. $Ag \rightarrow (\exists x)Bx$
   d. $(\exists x)[Cx \& (Dx \vee Lx)] \rightarrow (\forall x)Nx$
   e. $(\exists x)Ag$

2. For each of the following pairs, use the duality rules and Equivalences to translate the first member of the pair into the second member.
   a. $(\forall x)(Bx \rightarrow -Lx)$, $-(\exists x)(Bx \& Lx)$
   b. $-(\forall x)[Ax \rightarrow (\exists y)By]$, $(\exists x)[Ax \& (\forall y)-By]$

3. Translate into ordinary English, using the vocabulary provided.

   Vocabulary:
   Ax: *x* is an apple. *Gx*: *x* is good to eat. *Bx*: *x* is a banana. *Rx*: *x* is ripe.

   a. $(\forall x)[(Rx \& Bx) \rightarrow Gx]$
   b. $(\forall x)[Ax \rightarrow (Gx \rightarrow Rx)]$
   c. $-(\exists x)(Ax \& Bx)$
   d. $-(\forall x)(Rx \rightarrow Bx)$

4. Translate the following into symbolic notation using the vocabulary for the previous problem.
   a. Not all ripe bananas are good to eat.
   b. No apple is good to eat unless it is ripe.
   c. Some bananas are good to eat and some aren't.
   d. No apples are bananas.

5. Translate each of the following into symbolic notation, using the vocabulary provided.

   Vocabulary:
   *E*x: x is an employee. *R*x: x will receive a bonus. *F*x: x finishes early. *s*: Susan.

   a. If Susan finishes early, she'll receive a bonus.
   b. All employees will receive a bonus.
   c. If any employee finishes early, then Susan will receive a bonus.
   d. If Susan finishes early, then all employees will receive a bonus.

— 6 —

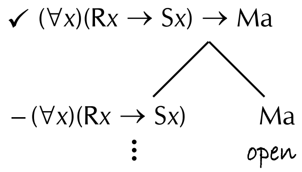
Quantification and Trees

6.1 Quantification and the Tree Method

All the usual rules for trees continue in force when quantified sentences appear in tree diagrams, but now some new rules must be added. The decomposition rules attempt to analyze sentences into their simplest elements. With respect to quantified sentences, this means getting rid of the quantifiers and instantiating any propositional functions. The process involves potential hazards against which you must constantly be on guard. The most important rule is this:

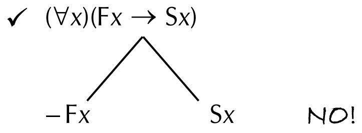
> ***Prime Directive:*** *Unquantified compound sentences (even if their components are quantified)* ***must*** *be decomposed in the usual manner, whereas quantified sentences cannot be decomposed until the quantifier is "dropped". (How that is done will be explained soon.)*

The sentence $(\forall x)(Rx \to Sx) \to Ma$ must be decomposed as usual because it is a non-quantified conditional sentence. It is true that its antecedent is a quantified sentence, but the sentence *as a whole* is not quantified, and so it must be decomposed in the way all conditionals are:

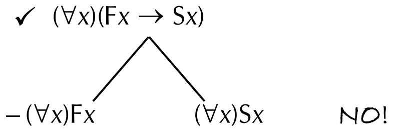


Since the tree's right branch is open, we needn't even bother with its left branch.

On the other hand, the sentence $(\forall x)(Fx \rightarrow Sx)$ cannot be decomposed yet, because it is a quantified sentence, and the Prime Directive says that it may not be decomposed until we somehow get rid of that quantifier. The following is therefore incorrect:



Notice two things. First, whoever made that tree violated the Prime Directive by trying to decompose without first somehow getting rid of the quantifier (because it governs the entire sentence). Second, some gibberish appears in both branches: $-Fx$ and Sx are both propositional functions, not bona fide sentences. (Remember, every variable, such as x, y and z, must appear with a matching quantifier.) Here is another incorrect attempt:



This is a not very clever attempt to decompose a quantified sentence without bothering to drop the quantifier first as required. No gibberish appears in the tree (that is, each variable has a matching quantifier), but the Prime Directive is violated. Somehow we must get rid of the quantifier. Sometimes that takes several steps. Any time we wish to drop a quantifier, we must first make sure this rule is satisfied:

> ***General Rule:*** *A quantifier may not be dropped unless the scope of the quantifier includes the* ***entire*** *sentence. This means that the quantifier must be on the extreme left hand side of the sentence (it cannot even be preceded by a denial sign) and the quantifier must govern the entire sentence. In order to get rid of an unwanted denial sign, apply the appropriate duality rule.*

What happens when a quantifier is "dropped" (and its propositional function instantiated) depends on whether it is a universal or an existential quantifier, as explained below.

6.2 Universal Instantiation

Examine this sentence: $-(\exists x)(Rx \& Sx)$. According to the Prime Directive, the sentence may not be decomposed, because it is a quantified sentence. So we would first have to drop the quantifier. But according to the General Rule the quantifier may not be dropped, because it does not govern the entire sentence: it is preceded by a denial sign. So the first step would be to get rid of the denial sign, which we could do by means of the quantifier duality rules.

✓ $-(\exists x)(Rx \ \& \ Sx)$
$(\forall x)-(Rx \ \& \ Sx)$

Now we would be in a position to deal with the resulting universal quantifier (because it governs the entire sentence) and then to decompose the rest of the sentence.

What are the rules for dropping quantifiers? There are two of them, one for universal quantifiers and one for existential quantifiers.

> ***Universal Instantiation (U.I.)****, a rule for dropping universal quantifiers: Rewrite the sentence without the quantifier, replacing every occurrence of its variable by an individual constant (i.e., a name) which occurs* ***anywhere*** *in that same branch (above or below). Do this instantiation once for each individual constant which occurs anywhere in that branch. If no individual names occur in the branch, then make one up. Although usually when a sentence is decomposed or otherwise dealt with, it is given a check mark, this case is the one and only exception: Do* ***not*** *give a check mark to the universally quantified sentence whose quantifier is dropped. (We'll see why later on.)*

Let's see how U.I. is used in a few cases, and then let's see how we might justify it. Suppose we are constructing a tree, and we have reached this point:

1. La
2. $(\forall x)(Hx \rightarrow Fx)$
3. Hs
4. $-Fs$

The Prime Directive says that we may not decompose the sentence in line 2 until we get rid of its quantifier. The quantifier is a universal quantifier which governs the entire sentence, and so we can apply U.I., which tells us to rewrite the sentence but without its quantifer. If that's *all* we did, the result would be, of course, this propositional function:

$$Hx \rightarrow Fx$$

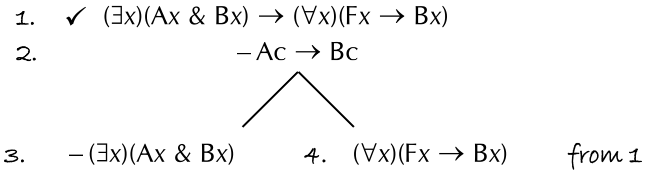
and that would be gibberish (because the variable *x* is governed by no quantifier) and not allowed in the tree. But the rule goes on: Replace every occurrence of the quantifier's variable in the resulting propositional function with a name. (That is, **instantiate** the propositional function.) Remember, a name, or constant, is *a*, *b*, *c*, etc. Which name should we use? U.I. tells us that we should use all the names which occur anywhere in that branch. That is, we instantiate once for each name. What names have occurred in the branch of our example above? *a* appears, and so we must instantiate with it. But *s* also occurs, and so we'll have to instantiate with *s* as well. And so on for all the names in the branch. (If *a* and *s* had not occurred—if there were no names in the branch—then we are instructed to invent any name we wish for the instantiation.) Since the universal quantifier was a quantifier over the variable *x*, we replace every occurrence of *x* in the propositional function (that is, the original sentence minus its quantifier) with an *a*. Then we do it *again* but with *s*. Here's what the tree would look like at that point:

1. La
2. $(\forall x)(Hx \rightarrow Fx)$
3. Hs
4. $-Fs$
5. $Ha \rightarrow Fa$ from 2 by U.I. using the a
6. $Hs \rightarrow Fs$ from 2 by U.I. using the s

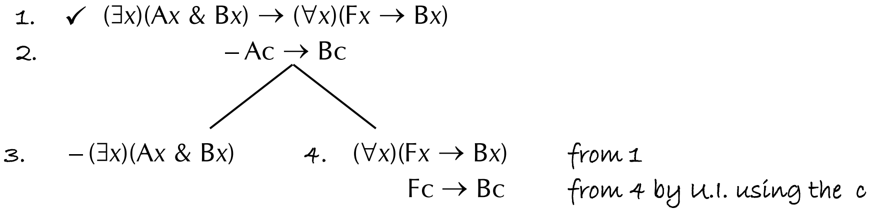
Let's try another example. Suppose we are constructing a tree and we have reached this point:

1. $(\exists x)(Ax \ \& \ Bx) \rightarrow (\forall x)(Fx \rightarrow Bx)$
2. $-Ac \rightarrow Bc$

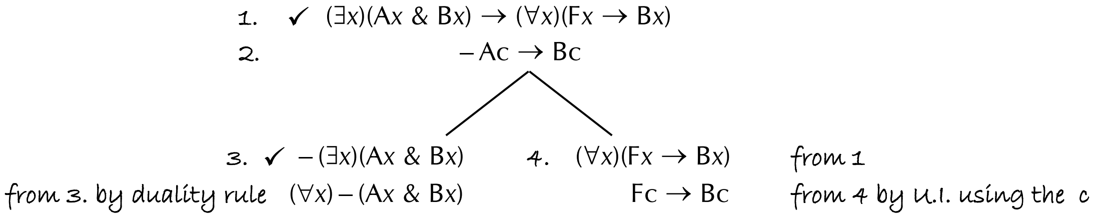
Clearly line 1 can be decomposed in the usual manner. In fact, the Prime Directive *requires* that if we want to deal with line 1, then we must decompose it in the usual manner, because it is not a quantified sentence; it is a regular conditional sentence (the parts of which just happen to be quantified). Let's decompose line 1 to see what happens.



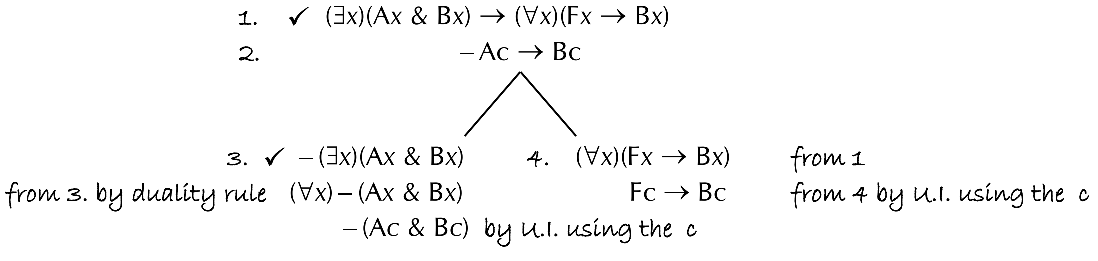
Now what do we do? We can decompose line 2, if we wish. Or we can deal with the quantifiers in the sentences on the last line. Let's do that, just for the fun of it. And let's start with the right branch. U.I. says to rewrite the sentence without its quantifier and replace each occurrence of its variable (the *x*) with names which occur in the branch. The name *c* occurs in the branch, and so whatever else we decide to do, we must use *c*:



(Notice that in accordance with U.I. the universally quantified sentence is not given a check mark when its quantifier is dropped.) Now the right branch will continue to be decomposed in the usual manner, the sentence in line 2 will be decomposed and placed under any open branches, etc. But now let's attend to the left branch. The General Rule says that the quantifier may *not* be dropped yet, because there is a denial sign in front of it. The Rule says that in such a case we are to apply the duality rules:



Now we can drop the quantifier on the resulting sentence. Using U.I., we must rewrite the sentence, replacing *x* with *c* (since *c* is a name which occurs in this branch):



(Notice that universally quantified sentences are not given check marks when their quantifiers are dropped.) And now the tree will continue in the usual manner.

How is Universal Instantiation (U.I.) justified? A universal quantifier claims that something is true of *all* things. For example, the sentence $(\forall x)Gx$ says that all things have the property *G*. Well, if it is true of all things, then it must surely be true for anything we can name: this chair, your Aunt Tillie, Mark Twain, your neighbor's stereo headphones, etc. Suppose some name, say *a*, occurs in a branch of the tree. If there is also a universally quantified sentence there, then what it says must surely hold for *a*. But if the name *b* occurs, then the universal claim holds for *b* as well. And so on for *all* names in that branch. And that is why, when dropping the universal quantifier, we rewrite it once for each name which occurs in that branch of the tree. (We don't go to other branches, because the consistency of a branch is independent of the consistency of any other branch.)

Another way to look at it is this: $(\forall x)Gx$ says that all things have the property *G*. We could rewrite that sentence as: "Aunt Tillie has the property *G*, and Mark Twain has the property *G*, and the neighbor's stereo headphones have the property *G*, and …." We won't finish this rewriting until we list everything in the universe. That's a big job. And that is why $(\forall x)Gx$ is a much more convenient way of expressing it. But when we are considering a branch of the tree, we don't really care about all things in the universe; we care only about all things mentioned in this branch of the tree, because each branch is independent of the others. (You can think of each branch as a possible universe.) Consequently, when we drop the universal quantifier and rewrite the sentence as a series of sentences, we must not omit anything in the universe—or rather the universe described by that branch of the tree.

There is another part to U.I., namely, that if no name occurs in the branch, then you must make one up. When you drop the quantifier, you must instantiate with some name or other; and if there is no name already in the branch, then you may make the rather weak (but metaphysically interesting) assumption that *something exists*, and you may give it an arbitrary name.

Here's another example. Test this argument for validity: "All humans are fallible. Therefore, some humans are fallible." This may be symbolized as: $(\forall x)(Hx \rightarrow Fx) \vdash (\exists x)(Hx \,\&\, Fx)$. We start the tree in the usual manner by listing the premise and the denial of the conclusion:

1. $(\forall x)(Hx \rightarrow Fx)$
2. $-(\exists x)(Hx \,\&\, Fx)$

Neither sentence may be decomposed, because both are quantified. We may drop the quantifier from the first sentence using U.I. Since no name appears yet in the branch, we assume that there is at least one thing in the universe; we can call it anything we want. Let's call it *a*:

1. $(\forall x)(Hx \rightarrow Fx)$
2. $-(\exists x)(Hx \,\&\, Fx)$
3. $Ha \rightarrow Fa$ from 1 by U.I. using a

(Remember not to give a check mark to the universally quantified sentence.) Now we can decompose line 3 as usual, but since that results in branching, let's deal with line 2. We may not drop the quantifer from that sentence, for two reasons: (1) we don't yet know the rule for dropping an existential quantifier; and (2) even if we did, we couldn't apply it until we obeyed the General Rule and got rid of that denial sign in front of the quantifier. So apply the duality rule to line 2 and we get:

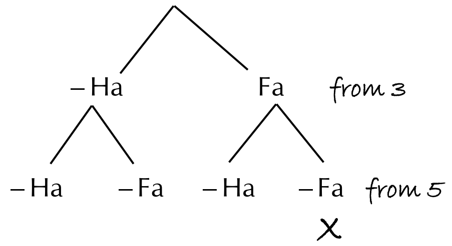
1. $(\forall x)(Hx \rightarrow Fx)$
2. ✓ $-(\exists x)(Hx \,\&\, Fx)$
3. $Ha \rightarrow Fa$
4. $(\forall x)-(Hx \,\&\, Fx)$ from 2 by duality rules

Notice that now we are dealing with a universally quantified sentence. (And that makes sense: the denial of an existentially quantified sentence is a universally quantified sentence; refer back to the diagram in section 5.6.) So let's apply U.I. to line 4. Since the name *a* occurs in this branch, we *must* instantiate with it. (And remember that when we use U.I., we do not give the sentence a check mark.)

1. (∀x)(Hx → Fx)
2. ✓ – (∃x)(Hx & Fx)
3. Ha → Fa
4. (∀x) – (Hx & Fx)
5. – (Ha & Fa) from 4 by U.I. using the a

Now we may decompose lines 3 and 5 in the usual manner.

1. (∀x)(Hx → Fx)
2. ✓ – (∃x)(Hx & Fx)
3. ✓ Ha → Fa
4. (∀x) – (Hx & Fx)
5. ✓ – (Ha & Fa)



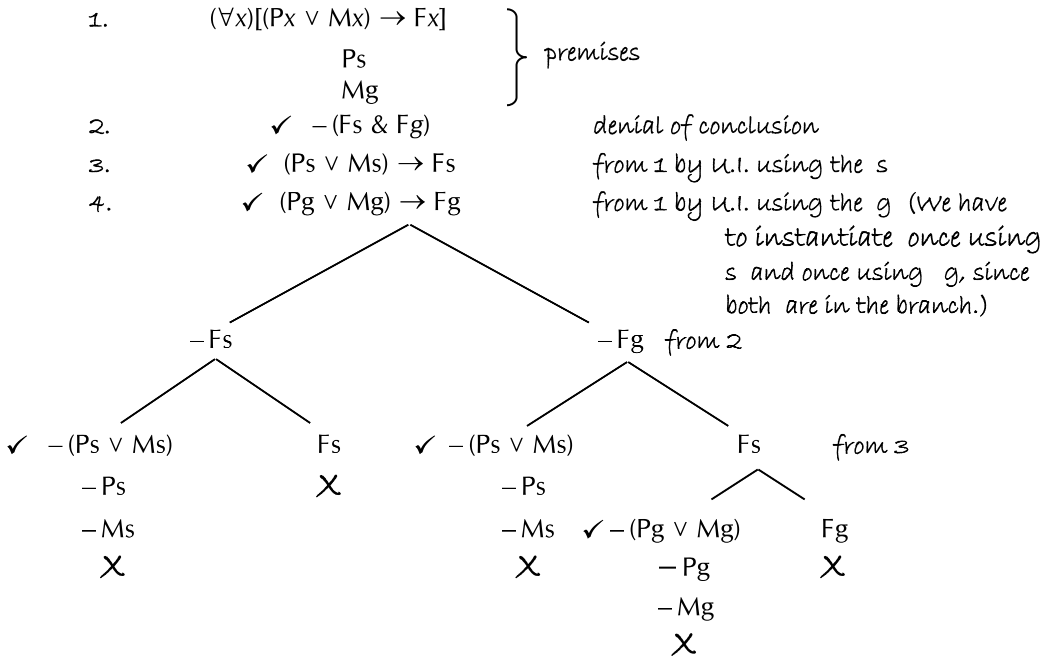
There is at least one open branch, and so the argument is not valid.

> *Aside:* Isn't that curious? Surely it *must* be valid. Surely if it is true that all humans are fallible, then it must also be true that some humans are fallible. If the argument were invalid, then it would have to be possible for the premise to be true and the conclusion false; that is, it would have to be possible for "All humans are fallible" to be true and yet "Some humans are *not* fallible" to be false. But that's absurd. Or is it? Let's pick out a counterexample from any open branch of the tree. The left-most branch will do. Suppose −*Ha* is true. Then the argument will have a true premise and a false conclusion. How shall we interpret −*Ha*? Simple: "Something named *a* is not human". What? How can that be a counterexample? Well, look more closely at the tree. What things are in the universe, according to the tree? At least one thing, named *a*. What else? *Maybe nothing!* The tree does not commit itself to the existence (or nonexistence, for that matter) of any other thing. Suppose the universe consisted of only one thing (call it *a*), and suppose that thing were not human. Then of course (∀*x*)(*Hx*→*Fx*) would be true, since nothing would satisfy the *Hx*, and a conditional with a false antecedent is true! In order for the argument to be valid, we would have to add another premise, something like: "There are humans". That is probably the premise you added tacitly when you first looked at the argument; if so, that is why the argument seemed to be valid. If you explicitly add the premise (∃*x*)*Hx*, then the argument will work. (Since we haven't yet discussed the rule for dropping existential quantifiers, you might try adding *Ha* to the premises instead.) The tree should then close.

Let's work another example. Use the tree method to test the validity of this argument:

(∀x)[(Px ∨ Mx) → Fx], Ps, Mg ⊢ Fs & Fg

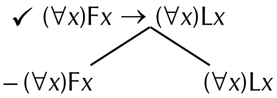
Try it on your own, and then compare your results to this tree:



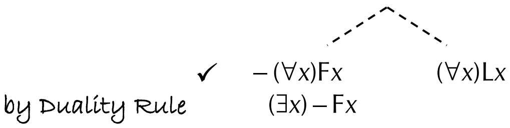
If we had mistakenly left out line 4 (instantiating on line 1 using the *g*), the tree would not have closed.

6.3 Existential Instantiation

Consider this sentence: (∀*x*)*Fx* → (∀*x*)*Lx*. Notice something very important: it is not a quantified sentence (although both its antecedent and consequent are quantified sentences). Since it is not a quantified sentence, the Prime Directive requires us to decompose it in the usual manner. (That is, we may *not* yet drop either of the quantifiers, because neither of them governs the *entire* sentence.)



Now let's look at the left branch. It cannot be decomposed, because it is a quantified sentence. So we'll have to drop its quantifier first. But we can't drop the quantifier, because it is preceded by a denial sign. So we apply the duality rule:



(∃*x*)–*Fx*, which is the last line in the left branch, says that there is at least one thing which does not have the property *F*. But which thing is it that fails to have that property? We are not told. In this branch, so far, we do not have the name of any particular thing. But we know that there *is* a particular thing, because the "(∃*x*)" guarantees it: "(∃*x*)" means "there exists at least one thing such that…". Let's give it an arbitrary name, say *a*. We can do this because we know that whatever the thing is, it is capable of having a name; and we also know that since nothing in this branch has a name yet, the name we choose won't conflict with the name of anything else. (If the name *a* had already occurred in this branch, then we would not be justified in giving this new thing the name *a*, because we would not be justified in assuming that this new thing is the same thing as that thing already named *a*, and so we would have to choose some new name for the new thing. If, later on, we found out that the new thing was the same thing as the previous thing, then we would learn that the two names were simply two names for the same thing, just as "Mark Twain" and "Samuel Clemens" are two names for the same thing. More on that in Chapter 7.) Thus, our tree continues in this way:

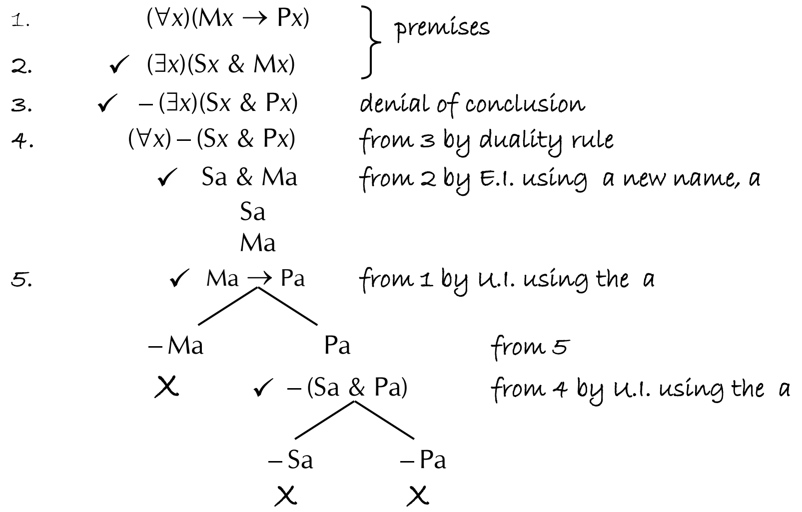
⋮
✓ (∃x) – Fx
– Fa by E.I. (see below) using the new name a.

Now we can state the rule which we have just discussed:

> ***Existential Instantiation (E.I.)****, a rule for dropping existential quantifiers: Under each open branch beneath the existentially quantified sentence, rewrite the sentence without its quantifier, replacing each instance of its variable with some* **new** *name, i.e., some name which does not yet appear in this branch. Whereas universally quantified sentences are not given check marks when their quantifiers are dropped, existentially quantified sentences are checked, because once they have been instantiated, we have given a name to the thing claimed to have the properties mentioned in the existentially quantified sentence, and we have no warrant for saying that a second thing also has those properties, and consequently we cannot use the sentence a second time. Remember, the existential quantifier says "there is* ***at least one*** *thing such that...." Whether there are two things, or three, or more, we don't know, and we are not allowed to make such presumptions.*

Here is an example using both U.I. and E.I. Translate this argument into symbolic notation and then use the tree method to test its validity:

> All mathematicians are philosophers. Some scientists are mathematicians. Therefore, some scientists are philosophers.

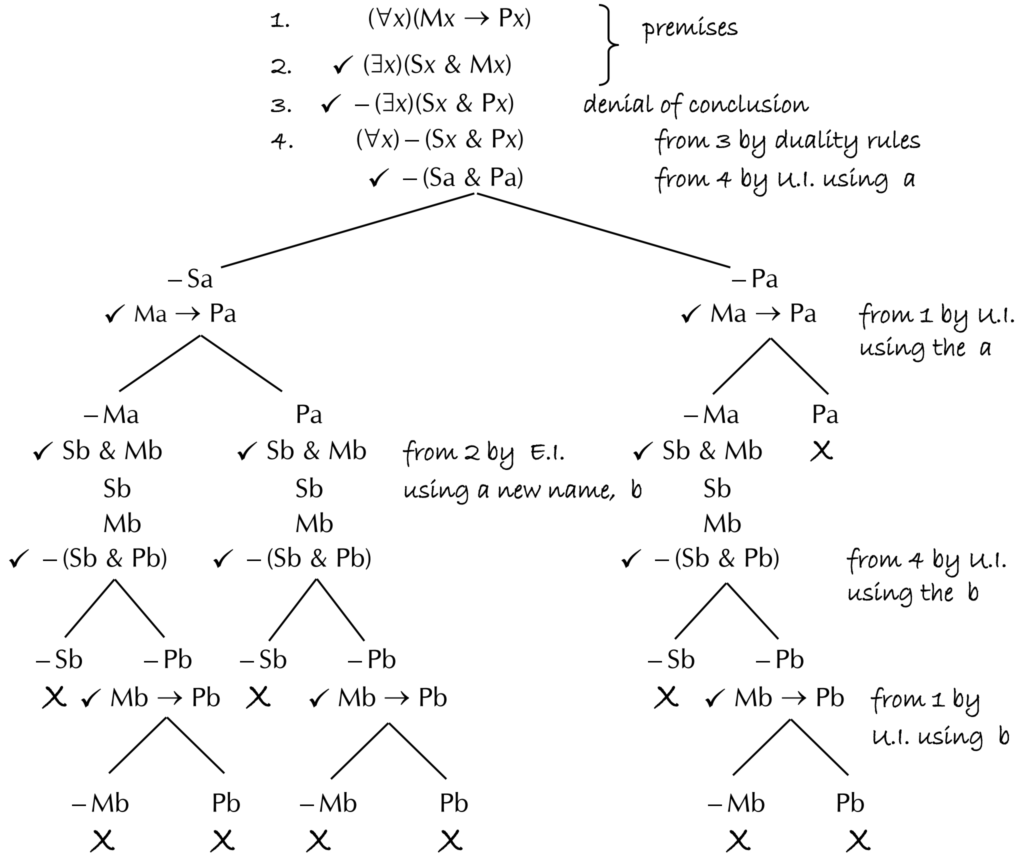


1. (∀x)(Mx → Px) } premises
2. ✓ (∃x)(Sx & Mx)
3. ✓ – (∃x)(Sx & Px) denial of conclusion
4. (∀x) – (Sx & Px) from 3 by duality rule
✓ Sa & Ma from 2 by E.I. using a new name, a
Sa
Ma
5. ✓ Ma → Pa from 1 by U.I. using the a
– Ma X | Pa from 5
✓ – (Sa & Pa) from 4 by U.I. using the a
– Sa X | – Pa X

A general warning: If you do not pay close attention to what you are doing, it is not hard to transgress the rules for dropping quantifiers.

Whenever there is a choice between dropping a universal quantifier and dropping an existential quantifier, it is a good strategic rule (but not logically required) to drop existential quantifiers first. If you do it the other way around, you will drop the universal quantifier and, in accordance with U.I., instantiate with names which appear in the branch. (If no name appears, you'll invent one.) And then when you drop the existential quantifier, you will not be able to use that same name; you'll have to use a new one. But now a new name appears in the tree, and so you'll have to go back and reapply U.I. to the universally quantified sentence, because U.I. says to instantiate with *every* name which appears in the branch, either above or below (even for names which appear in the branch at some later time, which is why a universally quantified sentence is not given a check mark when U.I. is used). If you do not go back and apply U.I. with the name which has newly appeared, some branches of the tree might seem to be open when they ought to have continued on to close off.

Here's how the previous tree will look if U.I. is done before E.I. The result is logically without fault, but the tree is bigger and more complex than it need be:



Exercise 6.1

* Answers to starred problems are given in Appendix D.

Use the tree method to test the following arguments. For any argument found invalid, give a counterexample.

* 1. All good philosophers are honored by their readers. All logicians are good philosophers. Hence, all logicians are honored by their readers.

2. Some people are doctors. Some people are lawyers. Therefore, some lawyers are doctors.

* 3. No Englishmen are Brazilian. No Brazilians are Hindus. Therefore, no Englishmen are Hindus.

4. Some rich men are fools. All rich men are loved. Therefore, some fools are loved.

* 5. No thief should expect to be let off easy. All people who do not make restitution to the victims after having taken possessions from the victims are thieves. Therefore, none of those who do not make restitution to the victims after having taken possessions from the victims should expect to be let off easy.

6. No true philosopher actively tries to suppress the truth. There are politicians who actively try to suppress the truth. Therefore, there are politicians who are not true philosophers.

7. There are some emotions which are not blameworthy. All emotions are passions. Therefore, there are some passions which are not blameworthy.

8. All the inhabitants of Mars are green. All green things blend in well with the Earth's environment. Therefore, some things which blend in well with the Earth's environment are inhabitants of Mars.

* 9. Only farmers are happy. There are rich men who are not farmers. Therefore, there are rich men who are not happy.

10. All who say you are an animal speak the truth. All who say you are a goose say you are an animal. Therefore, all who say you are a goose speak the truth!

11. $(\exists x)(Bx \,\&\, Sx),\ Ba \ \vdash\ Sa$
12. $(\forall x)(Ax \rightarrow Rx),\ Am \,\&\, Cm \ \vdash\ (\exists x)Rx$
*13. $Bj \rightarrow (\forall x)(Cx \rightarrow Ex),\ (\exists x)-Ex \ \vdash\ Bj \rightarrow -Cj$
14. $-(\forall x)(Cx \rightarrow Px),\ (\forall x)Px \rightarrow (\exists x)Ax \ \vdash\ (\exists x)Cx \rightarrow (\exists x)Ax$
*15. $Ra \rightarrow Rb,\ (\forall x)(Rx \rightarrow Cx) \ \vdash\ Ra \rightarrow Cb$

6.4 Infinite Branches

Construct a tree for this sentence: $(\forall x)(\exists y)(Mx \rightarrow Ry)$. Be careful to follow the General Rule: You may drop the universal quantifier, since it is on the far left of the sentence, but you may not (yet) drop the existential quantifier.

1.	$(\forall x)(\exists y)(Mx \rightarrow Ry)$	
2.	$(\exists y)(Ma \rightarrow Ry)$	from 1 by U.I. using a

Now the existential quantifier may be dropped (and a *new* name must be used to instantiate its variable):

1.	$(\forall x)(\exists y)(Mx \rightarrow Ry)$	
2.	✓ $(\exists y)(Ma \rightarrow Ry)$	from 1 by U.I. using a
3.	$Ma \rightarrow Rb$	from 2 by E.I. using a new name, b

Line 3 will decompose by branching, and neither branch will close. But we are clever, and we notice that the tree is not complete, for a new name (*b*) has appeared (in line 3), which means that the universal quantifier above in line 1 must be dealt with again, this time using that new name:

1.	$(\forall x)(\exists y)(Mx \rightarrow Ry)$	
2.	✓ $(\exists y)(Ma \rightarrow Ry)$	from 1 by U.I. using a
3.	$Ma \rightarrow Rb$	from 2 by E.I. using a new name, b
4.	$(\exists y)(Mb \rightarrow Ry)$	from 1 by U.I. using the b

But this has generated a new existentially quantified sentence, and so we must choose another new name when we drop its quantifier.

1.	$(\forall x)(\exists y)(Mx \rightarrow Ry)$	
2.	✓ $(\exists y)(Ma \rightarrow Ry)$	from 1 by U.I. using a
3.	$Ma \rightarrow Rb$	from 2 by E.I. using a new name, b
4.	✓ $(\exists y)(Mb \rightarrow Ry)$	from 1 by U.I. using the b
5.	$Mb \rightarrow Rc$	from 4 by E.I. using a new name, c

But now we'll have to deal with sentence 1 yet again, using the *c*. Obviously, this tree will go on forever, unless we can find some way of closing off the branches. But there is no way. What do we do? We will simply make it a rule that *a branch which can in no way be closed must therefore be an open branch*.

6.5 Further Considerations for Translations

Consider this sentence: "If a person leads a simple life, then he is happy". This is taken to hold for *all* persons, and we could just as well say, "People who lead simple lives are happy", or "Given any person, if s/he leads a simple life, then s/he is happy", or "All persons who lead simple lives are happy", etc. But it would be hasty to translate the sentence into symbolic notation as $(\forall x)(Lx \rightarrow Hx)$, because the variable x is a place holder for any nameable *thing* (not only persons), and the original sentence was obviously meant to apply specifically to persons. We can do one of three things in response to this translation problem.

First, we could simply let *Lx* stand for "*x* is a person who leads a simple life". This will do if we have no reason to distinguish things which are persons from things which lead simple lives. But we might have an occasion to make such a distinction if we wished also to talk about mere persons (whether they lead simple lives or not), or if we wished to discuss non-human things which lead simple lives.

A second possibility is to say "Given any thing, if it is a person and if it leads a simple life, then it is happy". In symbols: $(\forall x)[(Px \& Lx) \rightarrow Hx]$. (Or, by Exportation, $(\forall x)[Px \rightarrow (Lx \rightarrow Hx)]$, which translates as, "Given any thing, if it is a person, then, if it leads a simple life, then it is happy".) This method adequately solves the problem, although it means that we have to write an extra term (Px).

There is a third way. We may ***restrict the universe of discourse***. This means that we may stipulate that we are going to symbolize a sentence (or group of sentences, or an argument) such that the variables will not stand for things, but only for persons. That is to say, we will conceptually "squeeze down" the universe so that only persons are in it. That way, when we talk of anything, we will automatically be talking about the only things left in the universe, namely, persons. The terms "thing" and "person" will in such a case be synonymous. Note, however, that if we do restrict the universe of discourse to persons, we cannot also talk about cities, flowers, etc. in that same context (where by "context" we usually mean "argument"). If, for example, we wanted to symbolize the sentence, "Some people are foolish and some cities are large", then we cannot restrict the universe of discourse either to persons or to cities. In such a case we would leave the universe of discourse unrestricted and translate the sentence as: $(\exists x)(Px \& Fx) \& (\exists x)(Cx \& Lx)$.

Example: Is the following argument valid?

> Some large cities are not worth living in. Some small cities are worth living in. Therefore, some cities are worth living in and some are not.

If we symbolized the argument letting *Lx* stand for "*x* is a large city", *Sx* stand for "*x* is a small city", *Wx* stand for "*x* is worth living in", and *Cx* stand for "*x* is a city", then we would start the tree in this way:

$$(\exists x)(Lx \ \& \ -Wx)$$
$$(\exists x)(Sx \ \& \ Wx)$$
$$-[(\exists x)(Cx \ \& \ Wx) \ \& \ (\exists x)(Cx \ \& \ -Wx)] \qquad \text{denial of conclusion}$$

If you construct the tree, you will discover that it will not close. Yet the argument is surely valid, so we must have used inappropriate symbolization. After a bit of thought, it becomes evident that it is important to distinguish large things from small things from things which are cities. So let's adjust our vocabulary:

C*x* : *x* is a city
L*x* : *x* is large
S*x* : *x* is small
W*x* : *x* is worth living in

$$(\exists x)[(Cx \ \& \ Lx) \ \& \ -Wx]$$
$$(\exists x)[(Cx \ \& \ Sx) \ \& \ Wx]$$
$$-[(\exists x)(Cx \ \& \ Wx) \ \& \ (\exists x)(Cx \ \& \ -Wx)]$$

Now the tree ought to close. (Try it and see!)

On the other hand, we could announce that for this argument we will restrict the universe of discourse to cities. This will permit us to avoid writing *Cx*, since all the things we wish to talk about in this context are cities anyway. Try it that way, and you should discover that the tree will close.

You will have noticed by now that quantification is a very rich and useful method of symbolizing sentences. But there are many subtleties in English sentences which must be carefully noted in order to be symbolized correctly. Consider this sentence: "If anybody can solve the problem, Einstein can". This looks like a universally quantified sentence, but it is not. It does not say, "If *everybody* can solve the problem, Einstein can". Rather, it says something more like, "If there is at least one person who can solve the problem, then Einstein can". Restricting the universe of discourse to persons, we have:

(1) $(\exists x)Sx \rightarrow Se$

The meaning of the sentence is also expressed in another way: "If Einstein can't solve the problem, nobody can". Symbolically:

(2) $-Se \rightarrow (\forall x)-Sx$

Or, what is logically equivalent by the duality rules:

(3) $-Se \rightarrow -(\exists x)Sx$

Notice that your can get (3) from (1) (and vice-versa) by the Equivalence rule Contraposition.

In that example, only part of the sentence was quantified. Contrast that with this sentence: "If anyone can solve the problem, he is a genius". In symbols (once again restricting the universe of discourse to persons): $(\forall x)(Sx \rightarrow Gx)$. The clue to whether the whole sentence or only part of it is to be quantified in such cases usually takes the form of a pronoun—"he", "she", "it", "they", etc.—which refers back to the "anybody", "everybody", etc. found at the beginning of the sentence.

Exercise 6.2

* Answers to starred problems are given in Appendix D.

Symbolize these sentences. For convenience, assume that the universe of discourse is restricted to persons.

* 1. If anybody is a mathematician, he is trained in calculus.
2. If anybody is a good loser, Smith is.
* 3. If someone takes the initiative, he is rewarded.
4. If someone decides to sue, Beattie will be in hot water.
5. If everyone is a biologist, then I guess Einstein is too.

Consider this sentence: "Mathematicians and dentists are professionals". This cannot be translated as $(\forall x)[(Mx\&Dx) \rightarrow Px]$. Why not? Well, if you faithfully translate that symbolic version into English, you'll get: "If anyone is a mathematician *and* a dentist...", and that is of course not what the original sentence means. We should instead write:

(1) $(\forall x)(Mx \rightarrow Px)\ \&\ (\forall x)(Dx \rightarrow Px)$

Or we could write:

(2) $(\forall x)[(Mx \vee Dx) \to Px]$

(2) may be translated as: “If anyone is a mathematician or a dentist, then s/he is a professional.” And that certainly seems to capture the intent of the original sentence. As a matter of fact, (1) and (2) are logically equivalent, although it’s not obvious on the face of it. As an exercise, prove that they are. If they are equivalent, then they will always have the same truth values. (That’s the definition of logical equivalence.) If two sentences always have the same truth values, then their biconditional must be a tautology. That is,

$$[(\forall x)(Mx \to Px) \,\&\, (\forall x)(Dx \to Px)] \leftrightarrow (\forall x)[(Mx \vee Dx) \to Px]$$

must be a tautology. Use the tree method to test it. (If it is a tautology, the tree for its denial will close.)

Exercise 6.3

* Answers to starred problems are given in Appendix D.

Translate the following into symbolic notation.

1. Some valid arguments are reasonable.
2. Some invalid arguments are unreasonable.
* 3. The bat is a winged animal.
4. All swans that are not native to Australia are white.
5. Some apples are both green and ripe.
* 6. All fishes except sharks are kind to children.
7. Some students are neither diligent nor intelligent.
8. Only slithy toves gimble in the wabe.
9. Not every philosopher who is learned is wise.
* 10. No man is happy unless he is free.

Translate into ordinary English, using the vocabularies provided.

* 11. $(\forall x)[(Tx \,\&\, -Ax) \to -Lx]$
Tx: *x* is a triangle. *Ax*: *x* is equiangular. *Lx*: *x* is equilateral.

12. $(\forall x)[(Sx \,\&\, Wx) \to (Ex \,\&\, Px)]$
Sx: *x* is a secretary. *Wx*: *x* is well-liked. *Ex*: *x* is efficient. *Px* : *x* is pleasant.

* 13. $(\forall x)[(Rx \,\&\, Sx) \to Ex]$
Rx: *x* is a rectangle. *Sx*: *x* is a square. *Ex*: *x* has equal sides.

14. $(\forall x)[Gx \to (Px \leftrightarrow Cx)]$
Gx: *x* is a girl. *Px*: *x* is popular. *Cx*: *x* is cheerful.

* 15. $(\forall x)[(Gx \,\&\, Px) \to Cx]$
Gx: *x* is a girl. *Px*: *x* is popular. *Cx*: *x* is cheerful

16. $(\exists x)(Sx \,\&\, Dx) \to (\forall x)(Sx \to Bx)$
Sx: *x* is a student. *Dx*: *x* is diligent. *Bx*: *x* benefits.

17. $(\exists x)(Sx \,\&\, Dx) \to (\exists x)(Sx \,\&\, Bx)$
Sx : *x* is a student. *Dx*: *x* is diligent. *Bx*: *x* benefits.

18. $(\exists x)Mx \rightarrow (\exists x)Gx$
    Mx: *x* was murdered. *Gx*: *x* is guilty.

*19. $(\forall x)-Mx \rightarrow (\forall x)-Gx$
    Mx: *x* was murdered. *Gx* : *x* is guilty.

20. $(\exists x)(Sx \; \& \; Px) \rightarrow [(\forall x)(Dx \rightarrow Px) \vee (\exists y)(Dy \; \& \; -Hy)]$
    Sx: *x* is a lass. *Px*: *x* is promiscuous. *Dx*: *x* is a lad. *Hx*: *x* has a lass.

We have already come across sentences which involve nested quantifiers—that is, quantifiers within the scope of other quantifiers. (This is most often the case with *relations*, which we'll investigate in Chapter 7.) Consider this sentence: "For every ***p***roblem, there is a ***s***olution." Let *Px* stand for "*x* is a problem", and let *Sx* stand for "*x* is a solution". You might be tempted by several translations, all of them incorrect:

(1) $(\forall x)Px \rightarrow (\exists x)Sx$
(2) $(\forall x)Px \; \& \; (\exists x)Sx$
(3) $(\forall x)(Px \rightarrow Sx)$
(4) $(\forall x)(Px \; \& \; Sx)$

What's wrong with (1)? Upon examination, we see that it is a conditional, and so it ought to be roughly translated as "If $(\forall x)Px$, then $(\exists x)Sx$", which in turn is translated as "If everything is a problem, then there is some solution." But clearly the original sentence does not claim (even hypothetically) that *everything* is a problem! (2) is incorrect for a similar reason; it says that in fact everything is a problem (and, in fact, there is at least one solution). (3) has the form "All *P* is *S*", which is to say, "All problems are solutions." And (4) is even more obviously incorrect, once we translate it into English: "Everything is a problem *and* a solution." Clearly, none of (1) through (4) captures the meaning of the original sentence, "For every problem, there is a solution." What to do?

One solution to this problem involves using one quantifier within the scope of another:

(5) $(\forall x)[Px \rightarrow (\exists y)Sy]$

This may be translated in rather stilted language as, "Given any *x*, if it is a problem, then there is at least one solution." (Presumably, the solution is a solution to the given problem, although strictly speaking we cannot make the relationship between the problem and its solution explicit until we can symbolize relationships. That turns out to be a simple matter which we'll get to in the next chapter.)

The rules for dealing with quantifiers in trees apply to nested quantifiers. Nothing new is required, except for a little extra care in applying the rules properly and in correct sequence. Just for practice, let's test sentence (5) above to see if it is tautologous, contingent or contradictory. We begin with the universal quantifier, because it governs the entire sentence. After it has been dealt with, the inner details of the sentence are exposed, one of which is an existentially quantified sentence. And so on. If the tree closes, we know that the sentence is inconsistent. If it does not close, then it must be either contingent or tautologous, and so we'll have to construct a tree for the denial of the sentence to find out which. Here are the results:

1. $(\forall x)[Px \rightarrow (\exists y)Sy]$
2. ✓ $Pa \rightarrow (\exists y)Sy$ — from 1 by U.I., using a

$-Pa$ ✓ $(\exists y)Sy$ — from 2

open Sb — by E.I. using a new name, b

✓ $Pb \rightarrow (\exists y)Sy$ — from 1 by U.I., using the b

$-Pb$ ✓ $(\exists y)Sy$

open Sc — by E.I. using a new name, c

⋮

etc. (infinite branch)

There is at least one open branch, so the sentence is consistent. Now test the sentence's denial.

1. ✓ $-(\forall x)[Px \rightarrow (\exists y)Sy]$
2. ✓ $(\exists x)-[Px \rightarrow (\exists y)Sy]$ — from 1 by duality rules
3. ✓ $-[Pa \rightarrow (\exists y)Sy]$ — from 2 by E.I. using a new name, a

Pa — from 3

4. ✓ $-(\exists y)Sy$
5. $(\forall y)-Sy$ — from 4 by duality rules

$-Sa$ — from 5 by U.I. using the a

open

Since both the original sentence and its denial are consistent, the original sentence is contingent.

Exercise 6.4

* Answers to starred problems are given in Appendix D.

Translate and test for validity using the tree method. For any argument found invalid, provide a counterexample. First try your hand at the translations. If you need help, refer to the "Hints" further below.

* 1. Only persons have rights. HAL is a computer, not a person. Therefore, HAL has no rights.

2. Only actions which involve the initiation of force or fraud are unjust actions. Only unjust actions ought to be illegal. Prostitution is not an action which involves the initiation of force or fraud. Therefore, prostitution ought not to be illegal.

* 3. Anything which inhibits your ability to deal effectively with the circumstances in which you find yourself ought to be avoided. Drinking yourself into an insensitive stupor does that. Therefore, you ought to avoid drinking yourself into an insensitive stupor.

4. Einstein and Bohr were physicists. Physicists are metaphysicians only if they speculate on the ultimate nature of reality, which Einstein did and Bohr did not. Therefore, if Einstein was a metaphysician, then Bohr wasn't.

* 5. All dentists and monkeys are animals. Not all animals can extract teeth. No monkeys are dentists. Dentists can extract teeth. Therefore, monkeys cannot extract teeth.

6. Seymour is a robot and he can conduct the Philadelphia Symphony Orchestra. Robots are made of metal. Anything made of metal conducts electricity. Therefore, Seymour can conduct both electricity and the Philadelphia Symphony Orchestra.

7. Not all mammals are vegetarians. Therefore, not all vegetarians are mammals.

8. All mammals are warm blooded. No reptiles are mammals. Therefore, no reptiles are warm blooded.

* 9. Good science fiction writers are always creative. If anybody writes good science fiction, Larry Niven does. So Larry Niven is creative.

10. Egan's probability theory is probably incorrect. Anything which is probably incorrect is not absolutely certain and does not warrant our continued attention. If something doesn't warrant our continued attention, then it's unimportant. Hence, Egan's probability theory is unimportant.

11. Boris is a spy if and only if there are spies. There are indeed spies. So Boris is one of them.

12. All spies are secretive people. All people who are secretive have something to hide. So Boris, who is a spy, has something to hide.

13. Spies are not idiots. Boris is not an idiot. So Boris is not a spy.

14. Boris, like Natasha, is a spy. Some spies are treated as special cases at the border, and some spies aren't. Natasha is treated as a special case at the border. Therefore, Boris is not.

15. Only clever people are spies. So Natasha is clever, because she is a spy.

16. No arguments are valid. This is an argument. Therefore, this is not valid.

17. Some turtles are amphibians. Amphibians love to eat mice. Hence Billy-Bob is a turtle, because Billy-Bob is an amphibian and loves to eat mice.

18. Every patient in the waiting room is either sick or recovering. A patient in the waiting room is sick only if he/she has not yet seen the doctor. Not all patients in the waiting room are recovering. Hence, some patients in the waiting room are sick.

19. Every patient in the waiting room is either sick or recovering. A patient in the waiting room is sick only if he/she has not yet seen the doctor. Some patients in the waiting room have not yet seen the doctor. Hence, some patients in the waiting room are sick.

20. Peacocks are noisy but beautiful birds. Some peacocks are on postage stamps. Hence, some noisy birds are on postage stamps.

Hints:

1. The first premise says that being a person is a necessary condition for having rights.
   Suggested notation: *Cx: x* is a computer. *Rx*: *x* has rights. *Px*: *x* is a person. *h*: HAL.

2. Suggested notation: *F*x: *x* is an action which ... force or fraud. *O*x: *x* ought to be illegal. *P*x: *x* is prostitution (or an act of prostitution). *U*x: *x* is an unjust action.

   Note: You could also use *Jx* for "*x* is a just action", then symbolize "x is unjust" by $-Jx$.

3. Suggested notation: *I*x: *x* inhibits your ability, etc. *O*x: *x* ought to be avoided. *D*x: *x* is drinking yourself into an insensitive stupor.

   Note: The phrase "...does that" at the end of the second premise clearly means "...inhibits your ability, etc."

4. Suggested notation: e: Einstein. b: Bohr. *P*x: x is a physicist. *M*x: x is a metaphysician. *S*x: x speculates etc.

   Note: Here, as elsewhere, we make no logical distinction between "is", "was" and "will be". We interpret them all in a "tenseless" sort of "is".

5. Suggested notation: *D*x: *x* is a dentist. *M*x: *x* is a monkey. *A*x: *x* is an animal. *E*x: *x* can extract teeth

6. Take the first premise as two sentences, or as a conjunction of two sentences. Similarly for the conclusion. Be sure to translate the two phrases "conduct the P.S.O." and "conduct electricity" with two different letters.

9. The first premise may be taken to mean that all good science fiction writers are creative. The second premise is just another way of saying, "If anybody is a good science fiction writer, then...." And the word "so" is another word for "therefore".

10. Suggested notation: *A*x: *x* is absolutely certain. *W*x: *x* warrants our attention. *I*x: *x* is probably incorrect. *U*x: *x* is unimportant. e: Egan's probability theory.

Chapter 6 Test

1. Translate each of the following into ordinary English, using the vocabulary provided.

   Vocabulary:
   Lx : *x* is a liar
   Px : *x* is a politician
   g : Griswald
   Nx : *x* needs extra funding
   Sx : *x* seeks reelection

   a. $(\forall x)[Nx \rightarrow (-Lx \rightarrow -Px)]$
   b. $Lg \rightarrow -(\exists x)[(Px \,\&\, Sx) \,\&\, Nx]$
   c. $Pg \rightarrow (\forall x)(Sx \rightarrow Nx)$
   d. $(\exists x)[(Px \,\&\, Sx) \,\&\, Nx] \rightarrow [(Pg \,\&\, Sg) \,\&\, Ng]$

2. Translate each of the following into symbolic notation, using the vocabulary above.

   a. All politicians who seek reelection or who need extra funding are liars.
   b. Griswald is a lying politician only if he seeks reelection and needs extra funding.
   c. Griswald is a liar, but not all who lie are politicians.

d. If some politicians seeking reelection do not need extra funding, then not all politicians who need extra funding are liars.

3. Determine whether the following sentence is tautologous, contingent or contradictory.

   $(\exists x)(-Ax \ \& \ -Gx) \leftrightarrow -(\forall x)(-Gx \to Ax)$

4. Use the tree method to determine whether these two sentences are logically equivalent. (Careful!)

   $(\forall z)(Pz \to Ga)$, $(\forall z)Pz \to Ga$

5. Use the tree method to determine whether these two sentences are logically equivalent. (Careful!)

   $(\exists z)(Pz \ \& \ Ga)$, $(\exists z)Pz \ \& \ Ga$

6. Use the tree method to show that each of the following equivalences is true.

   a. *Distributivity of* $\forall$ *over &:* $[(\forall x)Px \ \& \ (\forall x)Qx] \equiv (\forall x)(Px \ \& \ Qx)$
   b. *Distributivity of* $\exists$ *over* $\vee$*:* $[(\exists x)Px \vee (\exists x)Qx] \equiv (\exists x)(Px \vee Qx)$

7. Determine whether the following arguments are valid. If an argument is invalid, provide a counterexample.

   a. $(\forall x)(Ax \to -Bx), (\forall x)(Bx \to Cx) \vdash (\forall x)(Ax \to -Cx)$
   b. $Ob \to (\exists x)(Mx \ \& \ Ex), (\forall x)(Ex \to -Gx), Mj \ \& \ Ob \vdash -Gj$

For each of the following arguments, translate it into symbolic notation and test for validity. If it is invalid, provide a counterexample.

8. A person is married if and only if he/she has a spouse. No dead persons have spouses. So some persons are not married.

9. If something is a cat, then it is not a dog. Everything in the house is a cat, but nothing in the house belongs to Fred. So Fred owns no dogs.

10. No cocarubbhas are bruharubbas. There are some bruharubbus, so not everything is a cocarubbha.

11. If there are UFOs, then all Army generals are liars. There are no lying Army generals, so not everything is a UFO.

12. There are no apple green Toyotas. There are no cars in this race that are not apple green. Hence, there are no Toyotas in this race.

13. Arthur often visits the local art gallery. No one who visits the local art gallery often but fails to contribute to its fund drive deserves to be acknowledged as a patron of the arts. Hence if Arthur does not deserve to be acknowledged as a patron of the arts, then he does not contribute to the local art gallery's fund drive.

14. All cowboys are loners. People who works with cows are not rich. Cowboys work with cows. So some loners are not rich.

15. Fortune smiles on only those who are rich and famous. Margaret is not rich. Therefore fortune does not smile on Margaret.

— 7 —
Relations

7.1 Relations

Right now we have no convenient way to symbolize many kinds of important relationships which might hold among things. Relations may be dyadic (between two things), triadic (among three things), and so on. For example, the relation expressed by "*A* is to the left of *B*" is dyadic. The relation expressed in "*A* is between *B* and *C*" is triadic. A relation involving four terms is exemplified by "Art sold Gwen a boat for one hundred thousand dollars".

Relational predicates can easily be added to our symbolic language. We will adopt a common (though not universal) convention of symbolizing the relation by means of a capital letter, as though it were a predicate. The things which stand in that relation are represented in the usual way with lower case letters (either names or variables), but we will string those letters out to the right of the relation symbol in the order they occur in the sentence. Thus, to symbolize "*A* is longer than *B*" we could let *L* stand for the dyadic relation "is longer than" and write *Lab*. "*D* is the father of *B*" could be symbolized as *Fdb*. A sentence involving a triadic relation is "*A* mistook *B* for *C*" and might be rendered as *Mabc*.

If you have some experience with mathematics or computer programming, you may think of the relation symbol—the capital letter—as the name of a function and the lower case letters as data given to that function. (Such data are often called "arguments"; but of course that term is used not in our sense of premises and conclusion.) The same holds for non-relational predicates as well; they are like functions which accept a single datum ("argument"). Thus a computer programming language might have a function for creating the square root of a number, and it might be written something like **sqrt(***x***)**, where *x* is the number you give to the function; the function then produces (or takes on the value of) the square root of *x*. Such a function is given only one number to work on, and so it resembles what we have been calling a predicate. In fact, you can think of predicates as single argument functions which calculate a value of T or F. Other functions might require multiple arguments. A function to calculate the greater of two numbers might be **greater(*x*, *y*)**. In our notation we could write Gxy. There might be a function **print(*c*, *x*, *y*)** to print a certain character, c, on the screen at coordinates x, y. In our notation, we might write $Pcxy$. And so on.

Exercise 7.1

* Answers to starred problems are given in Appendix D.

Translate the following into ordinary English, using this vocabulary:

> Mx: x is male. Pxy: x is the parent of y. Fx: x is female. Sxy: x is the sibling of y. a: Alice. e: Elizabeth. f: Fred. (And for convenience, assume the universe of discourse is restricted to persons.)

NOTE: Take both Fx and $-Mx$ to mean x is female, and both Mx and $-Fx$ to mean x is male. That is, assume $(\forall x)(Mx \oplus Fx)$.

1. Pfa
2. $Pfa \,\&\, Mf$
* 3. $(\exists x)(Pex \,\&\, Pxa)$
4. $(\exists x)(Pfx \,\&\, Pxa) \,\&\, Mf$
5. $Fe \,\&\, Sef$
6. $Mf \,\&\, Sfe$
* 7. $Mf \,\&\, -Sfe$
* 8. $-(\exists x)Pxa$
9. $-(\exists x)Sax$
10. $(\forall z)-Pzz$

Translate the following into symbolic notation.

*11. ***R***ochester is ***b***etween ***B***uffalo and ***S***yracuse.
*12. ***G***eorge is ***M***ary's ***e***mployer.
13. ***M***at is ***G***eorge's ***e***nemy.
14. ***A***lice is ***t***aller than ***S***igfried.
15. Alice is the same height as Sigfried.

When quantifiers are used with relations, it is not uncommon that two or more quantifiers will appear in a single sentence, and when that happens, we must pay careful attention to their placement. We have already seen in Chapter 6 that sentences can incorporate one quantifier within the scope of another, but now that will become much more common. Consider this rather simple claim about numbers: "For every number, there is a number greater than it." Letting G represent the relation "is greater than" (and, for convenience, restricting the universe of discourse to numbers), how should we symbolize the sentence? We can't write something like $(\forall x)Gxy$, because (for one thing) y would be a variable without a matching quantifier; we would need $(\forall y)$ or $(\exists y)$ in there somewhere. The solution is to string the two quantifiers together: $(\forall x)(\exists y)Gyx$. The quantifiers are given in the order in which they occur in the English sentence. If it seems somehow odd that two quantifiers should be concatenated in that way, then you may alternatively write: $(\forall x)[(\exists y)Gyx]$, which more clearly shows that everything within the square brackets lies within the scope of the universal quantifier.

Note that the *order* in which the quantifiers are given is important. What does this sentence say? $(\exists x)(\forall y)Gyx$. We translate it from left to right as usual: "There exists a number such that all numbers are greater than it." Clearly, this is meaningful, yet just as clearly false (it would imply that there is a smallest number), whereas the other sentence above was clearly true. The order in which the variables occur after the relation symbol is also important. Here are a number of variations produced by switching the order of the quantifiers and the order of the variables. Only three of the variations (4, 5 and 8) turn out to be true (on ordinary interpretations of numbers, anyway); the rest are false.

1. $(\forall x)Gxx$ All numbers are greater than themselves. (False.)
2. $(\exists x)Gxx$ There is some number which is greater than itself. (False.)
3. $(\forall x)(\forall y)Gxy$ All numbers are greater than all numbers. (False.)
4. $(\forall x)(\exists y)Gxy$ Given any number, it is greater than some number. (True.)
5. $(\forall x)(\exists y)Gyx$ Given any number, there is some number greater than it. (True.)
6. $(\exists x)(\forall y)Gxy$ There is some number such that it is greater than all numbers. (False.)
7. $(\exists x)(\forall y)Gyx$ There is some number such that all numbers are greater than it. (False.)
8. $(\exists x)(\exists y)Gxy$ There is some number which is greater than some number. (True.)

Notice in particular the difference between variations 4 and 6. In variation 6, reading from left to right in the usual way in translation, we first identify a particular number, and then we compare it to all numbers. The result is obviously a false claim. But in variation 4 there is some possible ambiguity. Are we comparing all numbers with some particular number? No; although we are considering all numbers, we should think of it as examining the numbers one at a time (and we continue doing this until we have examined all the numbers). That is, we pick a number (it can be completely at random) as an instantiation for the x in $(\forall x)$, do something with it, then go back and pick a new instantiation for $(\forall x)$, and so on. Now, each time we choose a new number, we *then* search for some particular number to compare it to, and that particular number need not be the same number each time we go through the $(\forall x)$. Think of it this way: The $(\forall x)$ says to pick any number. (We can choose at random, because the predicate or relation is claimed to hold for *all* numbers, which means it does not matter which number we choose.) Having picked a number, we then ask whether there is some number or other which is smaller than our chosen number. The answer is, of course, yes. Now we go back to the $(\forall x)$ and choose *another* number (at random), and of this new number we ask whether we can find some smaller number. Again the answer is, yes. But notice that the "some number" which is smaller in the first case need not be the same "some number" which is smaller in the second case. For example, start with a randomly chosen number—let's say 345. Is there some smaller number? Yes; 344 is one such a number, and we need only one to make the case. Now choose another number for the universally quantified x—say 15. Is there some smaller number? Yes; 10 is such a number (the 344 has already been discarded). And so on and on until the cows come home.

In variation 6, in contrast, there is said to be some number such that for all y it (the number) is greater than y. In order to determine the truth or falsity of the claim, we search about for some number with the property of being greater than all numbers. Once having fixed upon some particular number as a candidate, we hold that number constant and then proceed to compare it to all the numbers in the universe. Of course, we will find that our chosen number is not greater than all numbers. But perhaps we chose the wrong number; just because *this* number does not satisfy the conditions, it does not follow that no other number could not succeed. So we choose another candidate and proceed to compare it to *all* numbers. Once again we will fail. Eventually we will be convinced that such a number just cannot be found—it is false that there is such a number.

Exercise 7.2

* Answers to starred problems are given in Appendix D.

Using the vocabulary in Exercise 7.1, translate the following sentences into ordinary English.

* 1. (∃x)(∃y)(Pxf & Sxy & Pya)
2. (∀x)(∀y)(Pxy → –Pyx)
* 3. (∀x)–Pxx
4. (∀x)(∃y)(My & Syx)
* 5. (∃x)(∀y)(Fy → –Syx)
6. (∃x)(∀y)–Syx
* 7. (∃x)(Mx & Sxa)
8. Fa & (∃x)Sax
9. (∀x)(Fx → –Sxa)
10. (∃x)(Fx & Pfx)

The tree method requires no new rules in order to handle relations. There are, however, some new cautions for the use of the tree method rules which become especially important. In particular, it is important to drop quantifiers in the legal order, one at a time. Consider, for example, the sentence −(∀*x*)(∀*y*)(*Ax*→*By*). Neither quantifier can be dropped, because neither governs the *entire* sentence; we must first get rid of the denial sign. We do that by applying the Duality Rules:

1. ✓ –(∀x)(∀y)(Ax → By)
2. (∃x)–(∀y)(Ax → By) *from 1 by duality rule*

The result is an existentially quantified sentence, and so the existential quantifier must be dropped in accordance with the rule Existential Instantiation (E.I.):

1. ✓ –(∀x)(∀y)(Ax → By)
2. ✓ (∃x)–(∀y)(Ax → By)
3. –(∀y)(Aa → By) *from 2 by E.I. using a new name, a*

Now the duality rule can be applied to line 3, then E.I. can be applied to the result. And so on:

1. ✓ –(∀x)(∀y)(Ax → By)
2. ✓ (∃x)–(∀y)(Ax → By)
3. ✓ –(∀y)(Aa → By) *from 2 by E.I. using a new name, a*
4. ✓ (∃y)–(Aa → By) *from 3 by duality rule*
5. ✓ –(Aa → Bc) *from 4 by E.I. using a new name c*

Aa *from 5*
–Bc
open

Exercise 7.3

* Answers for starred problems are given in Appendix D.

Use the tree method to test the validity of the following arguments.

* 1. $(\forall x)(\exists y)(Gx \rightarrow Myx),\ Ga \ \vdash\ (\exists x)Mxa$
2. $(\forall x)(\forall x \rightarrow Kxx),\ Jd \ \vdash\ Kdd$
* 3. $(\forall x)[Rx \rightarrow (\exists y)Lyx], \ \vdash\ Rc \rightarrow Lec$
4. $(\forall x)(\forall y)(\forall z)[(Gxy \ \&\ Gyz) \rightarrow Gxz],\ (\exists x)Gax,\ (\exists x)Gxb \ \vdash\ Gab$
5. $(\forall x)Rxa \ \vdash\ (\exists x)Rxa$
6. $(\forall x)(\forall y)(Lxy \rightarrow Lyx),\ Lab \ \vdash\ Lba$
* 7. $Scl,\ (\exists x)Sxl \rightarrow Bl \ \vdash\ Bl$
8. $Scl,\ (\exists x)(Sxl \rightarrow Bl) \ \vdash\ Bl$
9. $(\forall x)(\forall y)[Cxy \rightarrow (\exists z)Hz] \ \vdash\ (\forall x)(\forall y)Cxy \rightarrow (\exists z)Hz$
10. $(\forall x)[(Lx \ \&\ (\exists y)Pxy) \rightarrow Qx],\ -Qc \ \vdash\ -(\exists x)Pcx$

11. Anyone who likes Abby also likes Greta. So Ziggy does not like Abby, because Ziggy does not like Greta. (Use L*xy*: *x* likes *y*.)

12. Somebody is taller than Igor. No one is taller than Frank. Hence, Frank is taller than Igor. (Use T*xy*: *x* is taller than *y*.)

* 13. Anyone who is taller than Igor is also taller than someone who is taller than George. So if Kim is taller than Igor, then there is someone who is taller than George.

14. Sally loves Pat. Hence, somebody loves Pat. (L*xy*: *x* loves *y*.)

* 15. Anyone who likes Iverson will help Gannett. Abelson will help no one except a friend of Vickers. No friend of Sildotter has Gannett for a friend. Therefore, if Vickers is a friend of Sildotter, Abelson does not like Iverson. (L*xy*: *x* likes *y*. H*xy*: *x* helps *y*. F*xy*: *x* is a friend of *y*. *a*: Abelson. *g*: Gannett. *i*: Iverson. *s*: Sildotter. *v*: Vickers.)

16. No one loves Pat. Hence, Sally does not love Pat.

17. Chimps are faster than snakes. Snakes are larger than beetles. Larger things are faster than anything they are larger than. So chimps are faster than some things which are faster than beetles. (C*x*: *x* is a chimp. S*x*: *x* is a snake. B*x*: *x* is a beetle. F*xy*: *x* is faster than *y*. L*xy*: *x* is larger than *y*.)

* 18. Alfred likes Betty. Whoever likes Betty likes Carl. Alfred likes only rich people. Hence, Carl is rich.

19. Donald loves anyone who loves Ellie. Anyone who loves Frances loves Ellie. Donald loves Frances. Hence Donald loves himself.

20. Some movie stars love only people who do not love being the center of attention. Some movie stars do not love being the center of attention. Hence, there are movie stars who are not beloved by any movie stars. (M*x*: *x* is a movie star. L*xy*: *x* loves *y*. C*x*: *x* loves being the center of attention.)

7.2 The Identity Relation

A very special relation which may hold between two things is the relation of identity. To say that two things are identical is to say that they are not two things at all, but actually only one.

> *Aside:* There is an interesting philosophical claim, associated with W. G. Leibniz (1646–1716), called the Principle of the Identity of Indiscernibles. Briefly stated, this principle requires us to treat two things, for which there are no discernible differences, as one thing. The principle is akin to the fundamental principle of pragmatism, espoused by C. S. Peirce (1839–1914), which says that our idea of something is just our idea of its effects, so that if we believe that two things always have the same effects, then it is foolish to claim that there are two different things. More colloquially, there can be no difference which cannot make a difference.

To say that a is identical to b is to say that the thing named by the name a is the very same thing named by the name b. The two names may be different, but there is only one thing; evidently b is an alias for a. In keeping with the manner in which we have been symbolizing other relations, we might symbolize the relation of the identity of two things, x and y, by Ixy. However, because identity is such an important relation, we will honor it with a special form: $x{=}y$, where "=" does not mean "is equal to", but rather "is identical to" or "names the same thing as". To deny $x{=}y$, we will write either $-(x{=}y)$ or else $x{\neq}y$.

The use of identity in trees is really quite straightforward. It allows us to substitute one name for another *in the branch in which the identity occurs*. (It is this fact which allows us to close off some branches which we would not otherwise be able to.) Consider the following simple tree; it is apparently open:

1.	a = b
2.	Ra
3.	– Rb

But the identity of a and b given in line 1 allows us to substitute a for b (and b for a) whenever and wherever we wish. Thus, we could rewrite Ra (line 2) as Rb and/or we could rewrite $-Rb$ (line 3) as $-Ra$. That is, if Ra is true, then Rb must be true, since $a{=}b$. And, similarly, if Rb is false then Ra must be false. Either way, the tree will now close:

1.	a = b		OR	1.	a = b	
2.	Ra			2.	Ra	
3.	– Rb			3.	– Rb	
	– Ra	from 1 and 3			Rb	from 1 and 2
	X				X	

In addition, we will take any sentence of the form $-(p{=}p)$ or $p{\neq}p$ to be self-contradictory. Any branch in which such a sentence appears *must* be closed off immediately.

Difference

The identity relation allows us to symbolize some claims about *different* things. To say that two things are different is simply to deny that they are identical. Consider, for example, these two sentences:

1. Ra
2. Lb

They say that the individual named *a* has the property *R*, and the individual named *b* has the property *L*. May we conclude that *a* and *b* are *different* individuals? No. They might be, but then again they might not be. It is only if the claim $-(a=b)$ is also included that we are able to say for sure that one thing is *R* and a *different* thing is *L*.

Exactly One

The sentence $(\exists x)Rx$ says: "at least one thing is *R*". How shall we symbolize the claim that *exactly* one thing—no more, no less—is *R*? Another way of saying that exactly one thing is *R* is to say that at least one thing is *R*, and nothing else is. That is, there is at least one thing which is *R*, and given any other thing (i.e., anything which is not identical to the first), it is not *R*. Thus:

(1) $(\exists x)\{Rx \ \& \ (\forall y)[-(x=y) \rightarrow -Ry]\}$

Or we might put it this way: "At least one thing is *R*, and given anything, if it is *R*, then it is really the same thing as the first thing." In symbols:

(2) $(\exists x)\{Rx \ \& \ (\forall y)[Ry \rightarrow (x=y)]\}$

You can see that (2) comes from (1) by Contraposition on the expression inside the square brackets.

Exactly Two

We can also count higher. To say that exactly two things are *R* is to say that two different things are *R*, and nothing else is:

$(\exists x)(\exists y)\{Rx \ \& \ Ry \ \& \ x\neq y \ \& \ (\forall z)[Rz \rightarrow (z=x \vee z=y)]\}$

Using this technique you can count as high as you wish, although, as you can see, the symbolic expressions quickly become very long-winded, and traditional mathematical notation is much to be preferred.

Exercise 7.4

* Answers to starred problems are given in Appendix D.

Translate the following into ordinary English, given the vocabulary in Exercise 7.1 above.

* 1. $(\exists x)(Pxa \ \& \ x\neq f)$
2. $(\forall x)(\forall y)(Pxy \rightarrow x\neq y)$
* 3. $(\forall x)(\forall y)[(Mx \ \& \ Fy) \rightarrow x\neq y]$
4. $(Paf \ \& \ Fa) \ \& \ (\forall x)(x\neq a \rightarrow -Pxf)$
5. $(Paf \ \& \ Fa) \ \& \ (\forall x)(Pxf \rightarrow x=a)$
6. $Mf \ \& \ (\forall x)(Mx \rightarrow f=x)$
* 7. $Mf \ \& \ -(\exists x)(Mx \ \& \ x\neq f)$
8. $(\exists x)\{(Mx \ \& \ Sxa) \ \& \ (\forall y)[(Sya \ \& \ My) \rightarrow x=y]\}$

9. $(\forall x)(\forall y)(Sxy \rightarrow x \neq y)$
10. $(\exists x)\{Fx \,\&\, (\exists y)(Fy \,\&\, (\forall z)[Fz \rightarrow (z = x \vee z = y)])\}$

Use the tree method to test the validity of the following arguments.

*11. $(\forall x)(Rx \rightarrow Gx),\ Ra,\ a = b \ \vdash\ Gb$
12. $(\exists x)(\forall y)[(Rxy \,\&\, x \neq y) \rightarrow Nxy] \ \vdash\ -Naa$
*13. $(\exists x)[Rx \,\&\, (\forall y)(Ry \rightarrow x = y)],\ Ra \ \vdash\ Rb \rightarrow a = b$
14. $e = g \ \vdash\ g = e$
15. $a = b \ \vdash\ (\exists x)(\exists y)(x = y)$

*16. All students will be graduates. There is at least one student. There will be at most one graduate. So there is exactly one student.

*17. Mr Rabbit is faster than Mr Tortoise. Only Ms Ocelot is faster than Mr Rabbit. Therefore, Ms Ocelot is faster than Mr Tortoise.

18. Mr Rabbit is faster than Mr Tortoise. Therefore, Mr Rabbit is not Mr Tortoise.

19. Only Ms Ocelot is faster than Mr Tortoise. Ms Ocelot is a mammal. Hence, only a mammal is faster than Mr Tortoise.

20. Mr Rabbit is faster than Mr Tortoise. No one is faster than himself. Therefore, Mr Rabbit is not Mr Tortoise.

7.3 Properties of Relations

Consider this argument:

> Alice is taller than Boris.
> Boris is taller than Caldwell.
> Therefore, Alice is taller than Caldwell.

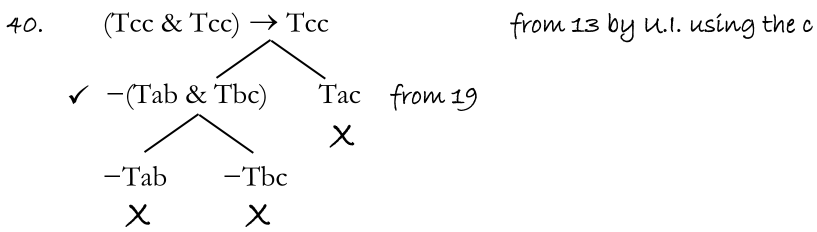
Translate it and test it using the tree method. The tree says that it is not valid. Yet the argument seems valid, doesn't it? What reasonable premise ought we to add to that argument in order to make it valid by the tree method? Unlike the argument presented on page 5–1 at the beginning of the discussion on quantification, where we realized that we needed more sophisticated tools, the failure here to prove that Alice is taller than Caldwell does not represent a deficiency in our symbolic apparatus. The problem, rather, lies in what we've *failed to make explicit*. And what is that? Perhaps some non-identities? We probably—and quite reasonably—assumed that Alice, Boris and Caldwell were three different persons. So perhaps we should make that explicit:

$a \neq b$
$b \neq c$
$a \neq c$

Unfortunately, this still does not cause the tree to close. So what else is there that we are assuming, but that has not yet been made explicit?

The problem has to do with what we failed to mention about the relation "is taller than": we failed to mention that the relation is ***transitive***—that if one person is taller than a second, and the second is taller than a third, then the first is taller than the third. If we add that general premise, then the tree will close. Here is the unnecessarily lengthy tree.

1.	$(\forall x)(\forall y)(\forall z)[(\mathrm{T}xy \,\&\, \mathrm{T}yz) \to \mathrm{T}xz]$	transitivity of relation T
	Tab	
	Tbc	
	$-\mathrm{Tac}$	denial of conclusion
2.	$(\forall y)(\forall z)[(\mathrm{Ta}y \,\&\, \mathrm{T}yz) \to \mathrm{Ta}z]$	from 1 by U.I. using the a
3.	$(\forall y)(\forall z)[(\mathrm{Tb}y \,\&\, \mathrm{T}yz) \to \mathrm{Tb}z]$	from 1 by U.I. using the b
4.	$(\forall y)(\forall z)[(\mathrm{Tc}y \,\&\, \mathrm{T}yz) \to \mathrm{Tc}z]$	from 1 by U.I. using the c
5.	$(\forall z)[(\mathrm{Taa} \,\&\, \mathrm{Ta}z) \to \mathrm{Ta}z]$	from 2 by U.I. using the a
6.	$(\forall z)[(\mathrm{Tab} \,\&\, \mathrm{Tb}z) \to \mathrm{Ta}z]$	from 2 by U.I. using the b
7.	$(\forall z)[(\mathrm{Tac} \,\&\, \mathrm{Tc}z) \to \mathrm{Ta}z]$	from 2 by U.I. using the c
8.	$(\forall z)[(\mathrm{Tba} \,\&\, \mathrm{Ta}z) \to \mathrm{Tb}z]$	from 3 by U.I. using the a
9.	$(\forall z)[(\mathrm{Tbb} \,\&\, \mathrm{Tb}z) \to \mathrm{Tb}z]$	from 3 by U.I. using the b
10.	$(\forall z)[(\mathrm{Tbc} \,\&\, \mathrm{Tc}z) \to \mathrm{Tb}z]$	from 3 by U.I. using the c
11.	$(\forall z)[(\mathrm{Tca} \,\&\, \mathrm{Ta}z) \to \mathrm{Tc}z]$	from 4 by U.I. using the a
12.	$(\forall z)[(\mathrm{Tcb} \,\&\, \mathrm{Tb}z) \to \mathrm{Tc}z]$	from 4 by U.I. using the b
13.	$(\forall z)[(\mathrm{Tcc} \,\&\, \mathrm{Tc}z) \to \mathrm{Tc}z]$	from 4 by U.I. using the c
14.	$(\mathrm{Taa} \,\&\, \mathrm{Taa}) \to \mathrm{Taa}$	from 5 by U.I. using the a
15.	$(\mathrm{Taa} \,\&\, \mathrm{Tab}) \to \mathrm{Tab}$	from 5 by U.I. using the b
16.	$(\mathrm{Taa} \,\&\, \mathrm{Tac}) \to \mathrm{Tac}$	from 5 by U.I. using the c
17.	$(\mathrm{Tab} \,\&\, \mathrm{Tba}) \to \mathrm{Taa}$	from 6 by U.I. using the a
18.	$(\mathrm{Tab} \,\&\, \mathrm{Tbb}) \to \mathrm{Tab}$	from 6 by U.I. using the b
19.	✓ $(\mathrm{Tab} \,\&\, \mathrm{Tbc}) \to \mathrm{Tac}$	from 6 by U.I. using the c
20.	$(\mathrm{Tac} \,\&\, \mathrm{Tca}) \to \mathrm{Taa}$	from 7 by U.I. using the a
21.	$(\mathrm{Tac} \,\&\, \mathrm{Tcb}) \to \mathrm{Tab}$	from 7 by U.I. using the b
22.	$(\mathrm{Tac} \,\&\, \mathrm{Tcc}) \to \mathrm{Tac}$	from 7 by U.I. using the c
23.	$(\mathrm{Tba} \,\&\, \mathrm{Taa}) \to \mathrm{Tba}$	from 8 by U.I. using the a
24.	$(\mathrm{Tba} \,\&\, \mathrm{Tab}) \to \mathrm{Tbb}$	from 8 by U.I. using the b
25.	$(\mathrm{Tba} \,\&\, \mathrm{Tac}) \to \mathrm{Tbc}$	from 8 by U.I. using the c
26.	$(\mathrm{Tbb} \,\&\, \mathrm{Tba}) \to \mathrm{Tba}$	from 9 by U.I. using the a
27.	$(\mathrm{Tbb} \,\&\, \mathrm{Tbb}) \to \mathrm{Tbb}$	from 9 by U.I. using the b
28.	$(\mathrm{Tbb} \,\&\, \mathrm{Tbc}) \to \mathrm{Tbc}$	from 9 by U.I. using the c
29.	$(\mathrm{Tbc} \,\&\, \mathrm{Tca}) \to \mathrm{Tba}$	from 10 by U.I. using the a
30.	$(\mathrm{Tbc} \,\&\, \mathrm{Tcb}) \to \mathrm{Tbb}$	from 10 by U.I. using the b
31.	$(\mathrm{Tbc} \,\&\, \mathrm{Tcc}) \to \mathrm{Tbc}$	from 10 by U.I. using the c
32.	$(\mathrm{Tca} \,\&\, \mathrm{Taa}) \to \mathrm{Tca}$	from 11 by U.I. using the a
33.	$(\mathrm{Tca} \,\&\, \mathrm{Tab}) \to \mathrm{Tcb}$	from 11 by U.I. using the b
34.	$(\mathrm{Tca} \,\&\, \mathrm{Tac}) \to \mathrm{Tcc}$	from 11 by U.I. using the c
35.	$(\mathrm{Tcb} \,\&\, \mathrm{Tba}) \to \mathrm{Tca}$	from 12 by U.I. using the a
36.	$(\mathrm{Tcb} \,\&\, \mathrm{Tbb}) \to \mathrm{Tcb}$	from 12 by U.I. using the b
37.	$(\mathrm{Tcb} \,\&\, \mathrm{Tbc}) \to \mathrm{Tcc}$	from 12 by U.I. using the c
38.	$(\mathrm{Tcc} \,\&\, \mathrm{Tca}) \to \mathrm{Tca}$	from 13 by U.I. using the a
39.	$(\mathrm{Tcc} \,\&\, \mathrm{Tcb}) \to \mathrm{Tcb}$	from 13 by U.I. using the b



The tree contains an exhaustive—and tedious!—use of U.I., whereas only lines 2, 6 and 19 were actually needed. A little attention to strategy would have avoided the unnecessary lines.

Transitive

What is this property of transitivity which relations may have? It is exemplified in the "is taller than" relation above; it is also found in many other relations: "is greater than", "is smaller than", "is equal to", "is smarter than", "is to the left of", and so on. If we let Φ stand for any relation symbol (i.e., it might stand for "T", "G", "S", etc., which in turn abbreviate the particular relations "is taller than", "is greater than", "is smarter than", etc.), then we may say that any relation Φ is transitive if and only if

$$(\forall x)(\forall y)(\forall z)[(\Phi xy \ \&\ \Phi yz) \rightarrow \Phi xz]$$

is true. If, for example, you replace Φ with "is greater than", then the above statement would be (if we restrict the universe of discourse to numbers): "If one number is greater than a second, and the second is greater than a third, then the first is greater than the third." And if we want to take that as a true sentence, then we shall say that "greater than" is a transitive relation.

Non-transitive

Many relations, such as "loves", "is a parent of" and "teaches" are not transitive. For example, even if Alice loves Boris and Boris loves Cindy, it does not mean that Alice loves Cindy (although she might, of course; it's just that the relation "loves" does not *require* it). So we may say that any relation Φ is ***non-transitive*** if and only if

$$-(\forall x)(\forall y)(\forall z)[(\Phi xy \ \&\ \Phi yz) \rightarrow \Phi xz]$$

is true of Φ. The relation "sees" is non-transitive, because if *A* sees *B*, and *B* sees *C*, then *A* might or might not see *C*. Notice how that symbolic version is formulated: the denial sign comes first. We are denying the transitive relation; we are, in effect, denying that the relation is always transitive, while allowing for the possibility that it might sometimes be. This might be more easily seen if we use the quantifier duality rules on the above expression to get an equivalent expression that says that you can find three things (x, y, and z) such that the relation Φ holds between x and y, and between y and z, but not between x and z:

$$(\exists x)(\exists y)(\exists z)[(\Phi xy \ \&\ \Phi yz) \ \&\ -\Phi xz]$$

Intransitive

In the first formulation of non-transitivity above, the denial sign came at the beginning; it simply denied, of some relation Φ, that it guarantied transitivity. Consider what it would mean to put the denial sign before the consequent:

$$(\forall x)(\forall y)(\forall z)[(\Phi xy \;\&\; \Phi yz) \rightarrow -\Phi xz]$$

This represents a much stronger condition. It says that if Φxy and Φyz, then it *cannot* be that Φxz. Such relations are called ***intransitive***. What relation could we substitute for Φ which would make such a claim true? If *A* is the one and only murderer of *B*, and *B* is the one and only murderer of *C*, then *A cannot* be the one and only murderer of *C*. So "is the one and only murderer of" is certainly an intransitive relation. The relation "is the mother of" might also be intransitive. If Alice is the mother of Betty (where by "mother" we mean "biological mother", and we exclude clever tricks such as including "grandmother" in the notion of "mother"), and Betty is the mother of Cindy, then surely Alice cannot be the mother of Cindy. At least not unless Alice and Betty could ever be the same person.* But, as was mentioned in section 2.2 on soundness, logicians are primarily interested in the logical relations among sentences, and not with whether contingent sentences really are true. We are therefore free to interpret the relation "is the mother of" in any way we see fit for the purposes at hand. Ordinarily, then, we would take the relation "is the mother of" to be intransitive, and not merely non-transitive. But if we were working out the logical consequences of time travel, perhaps we would have to entertain the possibility that "is the mother of" ought to be interpreted as non-transitive.

Reflexive

A relation Φ is said to be ***reflexive*** if and only if

$$(\forall x)\Phi xx$$

is true of it. Examples of reflexive relations are "is the same height as", "has the same mother as" and "is identical to", because anything is the same height as itself, has the same mother as itself, and is identical to itself.

Non-reflexive

Non-reflexive relations are relations which might sometimes hold between a thing and itself, and sometimes not. That is, a relation Φ is non-reflexive if and only if

$$-(\forall x)\Phi xx$$

is true of it. For examples, "talks to" (a person might—but need not—talk to himself), "makes fun of" (a person might—but need not—make fun of himself), and "is employed by" (a person might—but need not—be self-employed).

Irreflexive

Irreflexive relations are those which definitely *cannot* hold between a thing and itself. A relation Φ is irreflexive if and only if

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* But suppose time travel were possible. Alice and John give birth to Betty. Betty grows up, has a child—Cindy—then invents a time machine, travels back in time, changes her name to Alice and meets John. They have a child which they name Betty. Etc. Evidently Alice and Betty are in some sense the same person. If you're interested in this brain teaser, you will enjoy Robert Heinlein's short story, "All You Zombies—", in R.A. Heinlein, *The Unpleasant Profession of Jonathan Hoag* (Hicksville, NY, 1959).

$$(\forall x) - \Phi xx$$

is true of it. For examples, "taller than" (one cannot be taller than oneself), "is to the left of" (although something like this might be allowed if we accepted figurative claims such as "John was beside himself with joy", or if we wished to talk about strange universes wherein something could be in two places at once), and "is greater than" (a number cannot be greater than itself). Some relations seem at first to be irreflexive, but on further examination, they are somewhat problematic. Is "is the murderer of" irreflexive? Or is it, rather, non-reflexive? That is, is it possible to murder oneself? Or is suicide—the *killing* of oneself—not a form of murder? How about "deceives"? Can one deceive oneself?

Symmetric

Some relations are ***symmetric***, generally implying that the *order* in which the *relata* (the things related) are mentioned is unimportant. A relation Φ is symmetric if and only if

$$(\forall x)(\forall y)(\Phi xy \rightarrow \Phi yx)$$

is true of it. Examples would be "is next to" (if *A* is next to *B*, then *B* is next to *A*), "has a dialog with" and "is identical to".

Non-symmetric

Non-symmetric relations are those which might, but need not, relate the relata in either order. A relation Φ is non-symmetric if and only if

$$- (\forall x)(\forall y)(\Phi xy \rightarrow \Phi yx)$$

is true of it. If, for example, Alice loves Boris, there is no guarantee that Boris also loves Alice. Similarly, "is equal to or greater than" is non-symmetric.

Asymmetric

If a relation is *never* true for both arrangements of the relata, then the relation is said to be ***asymmetric***. A relation Φ is asymmetric if and only if

$$(\forall x)(\forall y)(\Phi xy \rightarrow - \Phi yx)$$

is true of it. "Is greater than" is such an asymmetric relation, because if *A* is greater than *B*, then surely *B* *cannot* be greater than *A*. "Is to the left of" would, in some contexts (such as on a two-dimensional plane, viewed from a fixed spot), be asymmetric, but in other contexts (such as on the surface of a sphere) it would be symmetric. "Is the mother of" (and some other blood relationships) are ordinarily interpreted as asymmetric.

Exercise 7.5

* Answers to starred problems are given in Appendix D.

* 1. Use the tree method to prove that the identity relation ("=") is transitive, reflexive and symmetric.

Indicate which one or more of the following nine categories apply to the relations below:

Categories: reflexive, non-reflexive, irreflexive, transitive, non-transitive, intransitive, symmetric, non-symmetric, asymmetric.

* 2. "is the brother of"
3. "logically implies"
4. "is (numerically) greater than"
* 5. "resembles"
6. "is physically contained in"
7. "scolds"
* 8. "surprises"
9. "is balanced precariously on top of"
10. "precedes"

Chapter 7 Test

1. Translate the following into symbolic notation, using the vocabulary provided.

Vocabulary:
Fx: *x* can fix the machine. *Mxy*: *x* has more experience than *y*. *g*: Gary. *s*: Susan.

a. Gary is the only one who can fix the machine.
b. Everyone except Gary can fix the machine.
c. If one person is more experienced than a second, and the second is more experienced than a third, then the first is more experienced than the third.
d. If Gary can fix the machine, then anyone with more experience than Gary can fix the machine.

2. Translate the following into ordinary English, using the vocabulary above.

a. $(\forall x)[Fx \rightarrow (\forall y)(Mxy \rightarrow Fy)]$
b. $(Msg \,\&\, Fg) \rightarrow Fs$
c. $Fg \rightarrow (\forall x)Fx$
d. $Fg \,\&\, Fs \,\&\, (\forall x)\{[-(x=g) \,\&\, -(x=s)] \rightarrow -Fx\}$

3. Use the tree method to determine whether these two sentences are logically equivalent.

$(\forall x)[Ax \rightarrow (\exists y)Rxy]$, $(\forall x)(\exists y)(Ax \rightarrow Rxy)$

4. For each of the following sentences, use the tree method to determine whether it is tautologous, contingent or contradictory.

   a. [(a = b) & (b = c)] → (a = c)
   b. (∀x)(∀y)Axy → (∃x)(∃y)Axy

5. Use the tree method to test the following arguments for validity. If an argument is invalid, give a counterexample.

   a. a = c, Ra, (∀x)(Rx → Tx) ⊢ Tc
   b. (∀x)(∀y)[(Rx & Dy) → Fxy], (∃x)(Rx & –Dx) ⊢ (∃x)(∃y)(Fxy & –Rx)

6. Translate and test for validity using the tree method. If the argument is invalid, provide a counterexample.

   Only Superman is faster than a speeding bullet. Anyone who can leap tall buildings in a single bound is faster than a speeding bullet. Clark Kent can leap tall buildings in a single bound. Hence, Clark Kent is Superman.

7. Translate and test for validity using the tree method. If the argument is invalid, provide a counterexample.

   Adrian has entered the race. So has William. Exactly one person has entered the race. Hence, Adrian and William are the same person.

8. Translate and test for validity using the tree method. If the argument is invalid, provide a counterexample.

   Exactly two persons have entered the race. Johnson is one of them. Either McCoy or Tremain is the other. It must be McCoy, because It's not Tremain.

9. Translate and test for validity using the tree method. If the argument is invalid, provide a counterexample.

   Anyone who is a friend of Peabody is a friend of Ahrenson. Any friend of Ahrenson is a friend of Peabody. Hence, Peabody is Ahrenson.

10. Translate and test for validity using the tree method. If the argument is invalid, provide a counterexample.

    Peabody has only one friend. All friends of Ahrenson are friends of Peabody. Hence, Ahrenson has only one friend.

— 8 —

Derivations with Quantification

8.1 Instantiation in Derivations

Universal Instantiation (U.I.) and Existential Instantiation (E.I.) were introduced back in Chapter 6, and they were used in connection with the tree method in both Chapter 6 and Chapter 7. Those rules were easy to justify. If, for example, $(\forall x)Px$ is true, then everything is P; and if everything is P, then everything that already has a name in the present context—an argument, say—is P, and if no particulars are mentioned in this context, then we allow ourselves to assume that something or other exists (i.e., that the universe is not empty) and it can be given an arbitrary name. So from $(\forall x)Px$ we may validly infer Pa (and Pb and Pc, etc. without limit). (We will extend this idea in a subtle way below.)

In the case of Existential Instantiation, we infer from $(\exists x)Px$ that Pa (only if a has not yet named anything in this context). That is to say, if it is true that there exists something that has the property P, then it can be given a name, and we must choose any name which has not already been used.

The two rules U.I. and E.I., used extensively with the tree method, can be used with the method of derivation as well, in a very obvious way. Prove the validity of the following argument, using the method of derivation:

All humans are mortal. Socrates is a human. Therefore, Socrates is mortal.

Proof:

1.	$(\forall x)(Hx \to Mx)$	
2.	Hs	$\vdash Ms$
3.	$Hs \to Ms$	1, U.I., using the *s*
4.	Ms	3, 2, Modus Ponens

As before, neither U.I. nor E.I. can be used unless the quantifier governs the *entire* sentence. If a quantified sentence is preceded by a denial sign, then you need to use one of the duality rules first; and if parts of a sentence are quantified, but the entire sentence is not, then you must use the Elementary Argument Forms or the Equivalences first. As a simple example, consider this problem: Prove: $(\exists x)(Hx \,\&\, -Mx) \to -La,\ Hb \,\&\, La \vdash Mb$.

1.	$(\exists x)(Hx \,\&\, -Mx) \to -La$	
2.	$Hb \,\&\, La$	$\vdash Mb$
3.	La	2, Separation
4.	$-(\exists x)(Hx \,\&\, -Mx)$	1, 3, Modus Tollens
5.	$(\forall x)-(Hx \,\&\, -Mx)$	4, Duality
6.	$-(Hb \,\&\, -Mb)$	5, U.I., using the *b*
7.	$-Hb \vee Mb$	6, DeMorgan's
8.	Hb	2, Separation
9.	Mb	7, 8, Disjunctive Syllogism

Notice that E.I. cannot be used on the sentence in line 1, because that sentence is a conditional sentence, not a quantified sentence (even though the sentence has a part that is quantified).

> **Aside:** Some writers have warned that our use of E.I. (for example, $(\exists x)Mx \vdash Mh$) should be taken as invalid, because there are easy counterexamples: From the acceptable claim that someone is merciful, it does not follow that Hitler is merciful.
>
> But I find this caution too hasty. It seems to me that the reason we are reluctant to allow "Hitler is merciful" to be a valid conclusion from "Someone is merciful" is that we already have in mind someone specific who is named "Hitler" (namely, the Nazi Führer), and it is *that* person we think of as being the *h* of the conclusion. And if that is so, then the name *h* is being thought of as already naming something in the context of this argument (even though the *h* does not actually appear until the conclusion). But the rule E.I. says to choose a *new* name; and in the context of the symbolic argument we are considering, the letter *h*, no matter what *h* abbreviates (whether Heathcliff or Hitler, or …), has not yet been used, and so, logically, we may not presume anything about the thing or person *we* have dubbed *h* (even if we have beliefs, outside of the present logical context, about a person named Hitler). And so there seems to be nothing logically amiss in $(\exists x)Mx \vdash Mh$.
>
> Oh, but there is something logically amiss after all. Although we used E.I. in the tree method, when we test rule E.I. itself by the tree method, the tree shows that it is invalid! (Try it with $(\exists x)Mx \vdash Mh$.) The invalidity of E.I., coupled with the obvious informal justification of it (that we are free to give a *new* name to the thing that is claimed, by $(\exists x)$, to exist) has led some logicians to offer a caution about E.I., namely, that it is invalid in itself, but when used within a larger argument, E.I. will never let us down. That is, E.I. will always be able to participate in a valid derivation, provided that the derivation does not *end* with E.I.
>
> That is a nice bit of logical maneuvering. But for our purposes, we will ignore the issues and go ahead and use E.I. (instantiating with *new* names, of course) without a worried conscience.

Exercise 8.1

* Answers to starred problems are given in Appendix D.

* 1. Fill in the justifications in the following derivation.

1.	Pa	$\vdash (\forall x)(Gx \to Pa)$
2.	$-(\forall x)(Gx \to Pa)$	______________
3.	$(\exists x)-(Gx \to Pa)$	______________
4.	$-(Gb \to Pa)$	______________
5.	$-(-Gb \vee Pa)$	______________
6.	$Gb \,\&\, -Pa$	______________
7.	$-Pa$	______________
8.	$Pa \,\&\, -Pa$	______________
9.	$-(\forall x)(Gx \to Pa) \to (Pa \,\&\, -Pa)$	______________
10.	$--(\forall x)(Gx \to Pa)$	______________
11.	$(\forall x)(Gx \to Pa)$	______________

2. Fill in the lines according to the justifications given.

1.	$(\forall x)(Ax \to -Cx) \quad \vdash (\forall x)(Wx \to Cx) \to -Aa$	
2.	Wa	
3.		C.A. (denial of concl.)
4.		3, Mat.Imp.
5.		4, DeMorgan's
6.		5, Separation
7.		5, Separation
8.		6, U.I. using the *a*
9.		8, 2, Mod. Pon.
10.		1, U.I. using the *a*
11.		10, 7, Mod. Pon.
12.		9, 11, Conjunction
13.		3→12, Cond. Proof.
14.		13, Reductio
15.		14, Double Neg.

Use the method of derivation to prove the validity of each of the following.

* 3. $(\forall y)[(Ay \,\&\, By) \to Cy],\ -Cb \ \vdash\ -(Ab \,\&\, Bb)$

4. $-(\exists x)Mx \ \vdash\ Mb \to Rb$

5. $(\forall x)[Ux \vee (\exists y)My],\ (\forall y)-Uy \ \vdash\ (\exists y)My$

8.2 Generalization in Derivations

Existential Generalization

Suppose that the conclusion of the longer argument above (p. 8–2) was not *Mb* but rather (∃*x*)*Mx*. How could we continue the argument after having reached step 9? Let us create a new rule, Existential Generalization (E.G.), that was not used in the tree method (because there was no use for it there). Informally, we may state and justify the rule this way:

> ***E.G.:*** *If a particular, named individual has a particular property, then we may infer the more general claim that* something *has that property—i.e., that there is at least one thing that has that property.*

If Bentham is mortal, then it must be true that something is mortal. If Bozeman is in Montana, then it must be true that something is in Montana. Etc. So our proof above would continue:

⋮	⋮	⋮
9.	Mb	7, 8, Disjunctive Syllogism
10.	(∃x)Mx	9, E.G.

Universal Generalization (U.G.)

Generalization does not succeed so easily in the case of universal quantification. For example, consider the previous argument again, and suppose we tried to generalize to the universal case based on line 10:

⋮	⋮	⋮
9.	Mb	7, 8, Disjunctive Syllogism
10.	(∃x)Mx	9, E.G.
11.	(∀x)Mx	10, U.G. <<< NO !

It should be obvious why line 11 fails: Line 10 claims that there is at least one thing that is *M*; but that cannot by itself justify the claim that everything is *M*. Nor can line 11 find justification in line 9, which says that *b* has the property *M*, because according to line 2, there is something else, *a*, in the universe, and we are not told whether *a* has the property *M*; if it does not, then it would be false that everything has the property *M*, which would contradict line 11. So line 11 is not deductively guaranteed on the basis of the previous lines. Of course, it *might* turn out that we could eventually prove that *a* does have the property *M*, but even if so, we would still not be justified in asserting line 11 until we were sure that *every* thing had the property *M*. If you are not quite sure of this, try testing this argument by means of the tree method: *Ma* ⊢ (∀*x*)*Mx*. You will find that the tree shows that the argument is invalid. Even if the premise does not identify the thing that has the property *M*, the argument will fare no better. Try this one: (∃*x*)*Mx* ⊢ (∀*x*)*Mx*.

Nevertheless, Universal Generalization does have a place; it can be justified in certain restricted conditions, namely, if there is a prior and special use of Universal Instantiation. The claim (∀*x*)*Mx* is the claim that everything is *M*. Another way of saying this is that any thing at all, chosen at random from the universe, is guaranteed to be an *M*. We make use of this fact in the tree method when we employ U.I., instantiating with all names mentioned in the present branch (i.e., in this “universe”) or else inventing a name, in case no name has appeared so far. It’s that step of inventing a name which concerns us here, because in the case of the tree method, it simply does not matter what new name we invent, because the tree method always continues in the direction of more particularity and never in the direction of more generality. As mentioned above, the tree method has no need for Existential Generalization. Similarly, it has no need for Universal Generalization.

But when it comes to the method of derivation, we can make good use of generalizations (both existential, as we saw above, and universal, as we’ll see below). What we need is a way to use Universal Generalization on those sentences involving particulars, but which are guaranteed to be universal in scope. We can find that guarantee in those sentences that have been produced *only* by Universal Instantiation. That is to say, we need a way to say something about the individuals—the particulars—that have been implied by universally quantified claims; and we need to keep track of those individuals so that we can generalize back to the universal claim.

Consider a simple argument which certainly looks valid:

> All politicians are liars. All liars disrespect the truth. Therefore, all politicians disrespect the truth.
>
> $(\forall x)(Px \rightarrow Lx),\ (\forall x)(Lx \rightarrow Dx)\ \vdash\ (\forall x)(Px \rightarrow Dx)$

You can use the tree method to verify that the argument is indeed valid. It looks like some form of Hypothetical Syllogism, but we have as yet insufficient tools in the method of derivation to derive the conclusion from those premises. If we use U.I. on the two premises, we can derive only a particular *instance* of the conclusion, and not the universal claim that the conclusion needs:

1.	$(\forall x)(Px \rightarrow Lx)$	
2.	$(\forall x)(Lx \rightarrow Dx)$	$\vdash (\forall x)(Px \rightarrow Dx)$
3.	$Pa \rightarrow La$	1, U.I., picking the name *a* at random
4.	$La \rightarrow Da$	2, U.I., using the *a*
5.	$Pa \rightarrow Da$	3, 4, Hypothetical Syllogism

Line 5 says that if *a* is a politician, then *a* disrespects the truth. But what about *b* and all the other possible things and persons in the universe? Without knowing about everything, we cannot generalize to the conclusion’s universal claim based on line 5’s claim about a particular individual.

What we need to understand is that in line 3 the *a* was an invented name; it did not appear earlier in the argument. What name we invented did not matter; since line 1 (which we were instantiating) is a universal claim, it must hold for anything and everything at random. Although U.I. requires that we use *some* particular name, we employed the rule on line 3 (and again on line 4) without caring whether the name was *a* or *b* or any other particular name. Let us make that attitude of “don’t care” explicit in the symbolic system itself. Let us show that we want to pick out an individual at random and therefore do not care which one it is. We can represent such a randomly chosen thing using not a variable (such as *x* or *y*, etc.), nor a constant (such as *a* or *b*, etc.), but instead a kind of “quasi-variable” (or “quasi-constant”*). Then, after we have been able to say something about this random thing, we can generalize back to the universal claim. We are justified in generalizing, because the thing that we chose was chosen completely at random, i.e., it could just as well have been any other thing.

(It is important to note that U.I. does not have to make use of such a quasi-variable; very often you will want to use U.I. to instantiate to a constant, as in the derivations above. But be on guard that when you do use a constant, you cannot then turn around and generalize back to a universal.)

Let’s use the symbol “*1*” (or “*2*” or any other number) to be the quasi-variable that will stand for any individual chosen at random from a universally quantified claim. “*1*” will mean “the thing that was chosen at random”. (We use “*1*” in order to allow for “*2*” and “*3*” and any other number just in case there are contexts in which we need to make multiple uses of U.I. to create random things that for some reason we want to distinguish from each other. But usually there will be no need to distinguish one randomly selected thing from another, and so usually our proofs will be easier to

---

* Or “varistant”? Or “constariable”?

construct if we use the same digit throughout. Usually.) Now let us see what happens to the derivation when we instantiate with a quasi-variable instead of a constant:

1.	$(\forall x)(Px \rightarrow Lx)$	
2.	$(\forall x)(Lx \rightarrow Dx)$	$\vdash (\forall x)(Px \rightarrow Dx)$
3.	$P1 \rightarrow L1$	1, U.I., using a quasi-variable, 1
4.	$L1 \rightarrow D1$	2, U.I., using a quasi-variable, 1
5.	$P1 \rightarrow D1$	3, 4, Hypothetical Syllogism

The derivation, so far, is almost the same as before except that we are guaranteed that the *1*, representing (as the constants *a* and *b*, etc. could not quite do) anything at random, can therefore be universally generalized. And, to emphasize something said above, notice that *1* can be used for U.I. from different universally quantified sentences. Line 1 refers to anything, and so does line 2; so in both lines 3 and 4 we help ourselves to the same random thing, which we temporarily call *1*. There would be nothing *logically* amiss in using *1* in line 3 and *2* in line 4 (just as, in the previous version, there would be nothing improper about using *a* for the U.I. in line 3 and *b* for U.I. in line 4). But doing so would be *strategically* inconvenient, because it would make getting to the conclusion more difficult.

Now we can complete the derivation using U.G.:

1.	$(\forall x)(Px \rightarrow Lx)$	
2.	$(\forall x)(Lx \rightarrow Dx)$	$\vdash (\forall x)(Px \rightarrow Dx)$
3.	$P1 \rightarrow L1$	1, U.I., using a quasi-variable, 1
4.	$L1 \rightarrow D1$	2, U.I., using a quasi-variable, 1
5.	$P1 \rightarrow D1$	3, 4, Hypothetical Syllogism
6.	$(\forall x)(Px \rightarrow Dx)$	5, U.G.

Line 6 makes use of the quasi-variable *1* in line 5, replacing *1* with a true variable (in this case *x*, but it could have been *y* or *z* or …) and quantifying over the entire resulting expression.

Nota bene: Quasi-variables are somewhere between true variables and constants. If U.I. can produce either quasi-variables or constants (as the occasion requires), then constants can be inferred from quasi-variables (but not vice versa!). So from $(\forall x)Px$ we can infer, by U.I., *P1*; we can also infer, again by U.I., *Pa*. And if it ever becomes necessary, we can infer both, which is also to say that $P1 \vdash Pa$ (but not vice versa!) is also acceptable.

Nota bene: To repeat a warning made earlier: The caution when using instantiation in the tree method also applies for the method of derivation, namely, that instantiation (whether universal or existential) is not allowed until and unless the quantifier governs the *entire* sentence. And now we can add a similar caution with regard to generalization: Generalization (whether universal or existential) is allowed only on the *entire* sentence. (Actually, there are some exceptions to this, but we will ignore them.) Consider, for example, this attempt at a derivation:

1.	$Pa \rightarrow La$	
2.	$(\exists x)Px \rightarrow La$	E.G. <<< NO !

The attempt was to generalize only the antecedent. The tree method will convince you that line 2 cannot be deduced from line 1. (The argument ought to be appreciated as invalid just by inspection: The premise claims that *if* Alfred is a politician, *then* he is a liar. The conclusion claims that if there exists at least one politician (it might be Reginald, for all we know), then Alfred (who might or might not be a politician) is a liar. Clearly, that conclusion does not follow from that premise.) If, however, E.G. is applied to the whole line, then the conclusion will be $(\exists x)(Px \rightarrow Lx)$ (which says that there is at least one thing such that if it is a politician, then it is a liar), and the argument will be valid (as you can verify with a tree).

There are some important restrictions on the use of Universal Generalization. We have noticed only one of them, but we need to examine another. Here they are:

1. ***U.G.*** *may be applied only in the case of quasi-variables "1", "2", etc. (because quasi-variables could come only from U.I., and we can generalize back to the universal case only if we know that the items we are generalizing about came from the universal case in the first place).*

2. ***U.G.*** *may not be used on an expression which contains a constant that was introduced either by E.I. or by a conditional assumption. (Other constants are not a problem.)*

What justifies the second restriction? Consider the following valid argument and its derivation.

> There is someone whom everyone admires. Therefore, everyone admires someone or other.
>
> $(\exists y)(\forall x)Axy \vdash (\forall x)(\exists y)Axy$

The derivation:

1.	$(\exists y)(\forall x)Axy$	$\vdash (\forall x)(\exists y)Axy$
2.	$(\forall x)Axa$	1, E.I., using a new name, *a*
3.	$A1a$	2, U.I., using a quasi-variable, *1*
4.	$(\exists y)A1y$	3, E.G.
5.	$(\forall x)(\exists y)Axy$	4, U.G.

Now consider a slightly different (and invalid!) argument:

> Everyone admires someone or other. Therefore there is someone whom everyone admires.
>
> $(\forall x)(\exists y)Axy \vdash (\exists y)(\forall x)Axy$

The attempted derivation:

1.	$(\forall x)(\exists y)Axy$	$\vdash (\exists y)(\forall x)Axy$
2.	$(\exists y)A1y$	1, U.I., using a quasi-variable, *1*
3.	$A1a$	2, E.I., using a new name, *a*
4.	$(\forall x)Axa$	3, U.G. <<< NO !
5.	$(\exists y)(\forall x)Axy$	4, E.G.

Line 4 is illegal because it uses universal generalization on an expression containing a constant (*a*) introduced by E.I. (in line 3). If you want to be especially cautious when using E.I. in order to avoid this kind of mistake, then you can attach some sort of reminder to the new constant introduced by E.I. (and similarly for a new constant introduced in a Conditional Assumption). For example, instead of *A1a* in line 3 above, you could have *A1a^*, where the "^" (or some other useful mark) might remind you that the constant *a* came from E.I., so you might be less likely to try to apply U.G to *A1a^*.

Exercise 8.2

* Answers to starred problems are given in Appendix D.

Use the method of derivation to prove the validity of each of the following.

A. Easy

* 1. $(\forall x)(Ax \rightarrow Bx),\ (\exists x)-Bx \ \vdash\ -Aa$
2. $(\forall x)[Ax \rightarrow (\forall y)By],\ Ac \ \vdash\ Bg$
3. $(\exists x)Mx \vee (\exists x)Nx,\ (\forall x)-Mx \ \vdash\ (\exists x)Nx$
4. $(\exists x)(Kx \ \&\ Lx) \ \vdash\ (\exists x)Kx$

* 5. $(\forall x)(Mx \rightarrow Lx),\ -(\forall x)(Rx \rightarrow Lx) \ \vdash\ (\exists x)(Rx \ \&\ -Mx)$

B. Medium Difficulty

* 6. $(\forall x)(Rx \rightarrow -Sx) \ \vdash\ -(\exists x)(Rx \ \&\ Sx)$
7. $(\forall x)(Ax \rightarrow Bx),\ (\forall y)(By \rightarrow Cy) \ \vdash\ (\forall z)(-Cz \rightarrow -Az)$
8. $(\forall x)(Sx \rightarrow Wx),\ (\forall x)(Yx \rightarrow Sx),\ (\exists x)-Wx \ \vdash\ (\exists x)-Yx$
9. $(\forall x)(Px \rightarrow Qx),\ (\exists x)-Qx,\ (\forall x)(-Px \rightarrow Mx) \ \vdash\ (\exists x)Mx$

* 10. $(\forall x)(Ax \rightarrow -Cx),\ (\forall y)(By \vee Cy),\ (\forall x)Ax \ \vdash\ (\forall y)By$

C. Higher Difficulty

11. $(\exists x)Dx \rightarrow (\exists y)Hy,\ (\forall x)(Mx \rightarrow Dx) \ \vdash\ (\exists x)Mx \rightarrow (\exists y)Hy$

* 12. $(\forall x)(Lx \rightarrow Mx),\ (\exists x)Lx \vee (\exists x)Mx \ \vdash\ (\exists x)Mx$
13. $(\forall x)[(Cx \ \&\ Zx) \rightarrow Ex],\ (\exists x)(Rx \ \&\ Zx),\ (\forall x)(-Cx \rightarrow -Rx) \ \vdash\ (\exists x)(Ex \ \&\ Rx)$

* 14. $(\forall x)[(Bx \vee Cx) \rightarrow Sx] \ \vdash\ (\forall x)[(Cx \ \&\ -Ax) \rightarrow Sx]$
* 15. $-(\exists x)(Mxa \ \&\ -Oxb),\ -(\exists x)(Dxc \ \&\ Dbx),\ (\forall x)(Oex \rightarrow Dxf) \ \vdash\ -(Mea \ \&\ Dfc)$

D. Good Luck

16. $(\exists x)Ex \rightarrow (\exists x)Ox \ \vdash\ (\exists y)(\forall x)(Ex \rightarrow Oy)$
17. $(\forall x)[(Rx \vee Ax) \rightarrow Tx],\ (\exists z)-(Mz \vee -Az) \ \vdash\ (\exists x)Tx$

* 18. $(\exists x)Gx \rightarrow (\exists x)(Ax \ \&\ Kx),\ (\exists x)(Kx \vee Lx) \rightarrow (\forall x)Mx \ \vdash\ (\forall x)(Gx \rightarrow Mx)$
19. $(\forall x)[(Fx \vee Gx) \rightarrow (Hx \ \&\ Kx)],\ (\forall x)\{(Hx \vee Lx) \rightarrow [(Hx \ \&\ Nx) \rightarrow Px]\}$
$\vdash\ (\forall x)[Fx \rightarrow (Nx \rightarrow Px)]$

* 20. $(\forall x)[Zx \rightarrow (\exists y)(Zy \ \&\ Rxy)],\ (\exists x)\{Zx \ \&\ (\forall y)[(Zy \ \&\ Rxy) \rightarrow Cxy]\} \ \vdash\ (\exists x)(\exists y)[(Zx \ \&\ Zy) \rightarrow Cxy]$

8.3 Identity and Derivation

There is nothing new when it comes to the use of identities in derivations, except for a need to invent some useful label when making use of an identity. We'll use "ID", along with two line numbers, one being the claim of identity, and the other containing the expression to substitute an identical name into. Here is a simple example.

1.	Rnf	
2.	n = s	
3.	Qs	⊢ (∃x)(Qx & Rxf)
4.	Qn	3, 2, ID
5.	Qn & Rnf	4, 1, Conjunction
6.	(∃x)(Qx & Rxf)	5, E.G.

Exercise 8.3

* Answers to starred problems are given in Appendix D.

Use the method of derivation to prove the validity of each of the following.

1. Rb & (∀x)(Rx → x = b), (∃x)(Rx & Sx) ⊢ Sb
* 2. (∀x)(Ax → Bx), (∀x)(Bx → Mx), Aa & −Mb ⊢ a≠b
3. Rd, (∀x)[Rx → (∃y)Myx], (∀y)−Mya ⊢ d≠a
4. (∃x)[Qx & (∀y)(Qy → y = x) & Rx], −Ra ⊢ −Qa
5. (∃x)(∀y)[(−Sxy → x = y) & Ax] ⊢ (∀x)[−Ax → (∃y)(y≠x & Syx)]
6. (∃x){Bx & [(∀y)(By →y = x) & Sx]}, (∃x)−(−Bx ∨ −Nx) ⊢ (∃x)(Nx & Sx)
* 7. (∀x)(Tx → x = a), (∀x)(Ux → x = b), (∃x)(Tx & Ux) ⊢ a = b
8. (∀x)(Mx → Vx), (∀x)(∀y)[(Vx ∨ Vy) → x = y], Mb ⊢ a = c

9. Heinrich is the one and only president of the company. But Freddy is a president of the company. Hence, Heinrich is Freddy.

10. There is exactly one president of the company. Hence, it's false that there are at least two presidents of the company.

* 11. There is exactly one president of the company, and he/she is a an excellent tennis player. Annie is not an excellent tennis player. So Annie is not the president of the company.

12. Exactly one pencil is on my desk, and it is yellow. There is at least one wooden pencil on my desk. Hence, something wooden is yellow.

* 13. Some art thieves have no sense of humor. Only Crown is an art thief. Only Thomas has no sense of humor. Therefore, Thomas is Crown.

Chapter 8 Test

Use the method of derivation to demonstrate the validity of the following arguments.

1. $(\exists x)(Ax \,\&\, Bx),\ (\forall y)(Ay \rightarrow Cy) \ \vdash\ (\exists x)(Bx \,\&\, Cx)$
2. $(\exists x)Mx,\ (\forall x)(-Sx \rightarrow -Mx),\ (\forall x)Qx \ \vdash\ (\exists x)Sx \,\&\, (\exists x)Qx$
3. $-(\exists x)(Mxa \,\&\, -Oxb),\ -(\exists x)(Dxc \,\&\, Dbx),\ (\forall x)(Oex \rightarrow Dxf) \ \vdash\ -(Mea \,\&\, Dfc)$
4. $(\forall x)[(Sx \,\&\, Bx) \rightarrow Ux],\ Be,\ (\forall x)Sx \ \vdash\ Ue \,\&\, Se$
5. $(\forall x)(Hx \rightarrow -Lx),\ (\exists x)(Nx \,\&\, Lx) \ \vdash\ (\exists x)(Nx \,\&\, -Hx)$
6. $(\forall x)(Kx \rightarrow -Lx),\ (\exists y)(Sy \,\&\, Ly),\ (\forall z)[(-Kz \,\&\, Sz) \rightarrow Qz] \ \vdash\ (\exists x)Qx$
7. $-(\forall x)Sx \ \vdash\ (\exists x)(Sx \rightarrow Ox)$
8. $-(\forall x)-(Cx \,\&\, Dx),\ (\forall y)(y \neq a \rightarrow -Cy),\ (\forall z)-(z \neq b \,\&\, Dz) \ \vdash\ a = b$

For each of the following arguments, translate it into symbolic notation and then derive the conclusion from the premises.

9. Doctor Who is actually Superman. Someone is both powerful and clever, but only Doctor Who is clever, and no one other than Superman is powerful.
10. Barnstormers and birds fly around in the sky. Anything that either flies around in the sky or taxis down the runway is not a product of nature. Hence, no barnstormers are products of nature.
11. There is no such thing as a mermaid, so if Trixie is a mermaid, then she is President of the Universe.
12. Mugsy, who is a contract killer, is an alias for Michael O'Hara. All contract killers live very short lives. Hence, Michael O'Hara will live a short life.
13. All lost works of art are valuable. All things in museums have been found. Therefore, if all valuable things are in museums, then all lost works of art have been found.
14. Only Superman is faster than a speeding bullet. Anyone who can leap tall buildings in a single bound is faster than a speeding bullet. Clark Kent can leap tall buildings in a single bound. Hence, Clark Kent is Superman.
15. Everything is either a square or triangle. Every square is a rectangle. There are non-rectangles. Therefore, there are triangles.

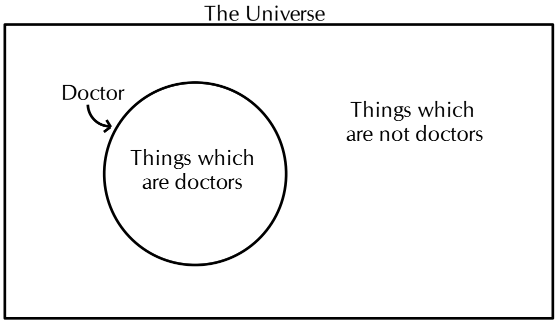
— A —

Appendix A: Venn Diagrams

The logic diagrams we will be discussing below come from the work of the British logician John Venn (1834–1923). A Venn diagram is a graphical method of representing some kinds of quantified sentences, both existentially and universally quanitified. Venn diagrams have severe limitations, but they nevertheless help to illustrate how predicates and variables are used to represent quantified expressions. This Appendix might profitably be read in connection with the study of Chapter 5 on Quantification.

A.1 Diagrams with One Term

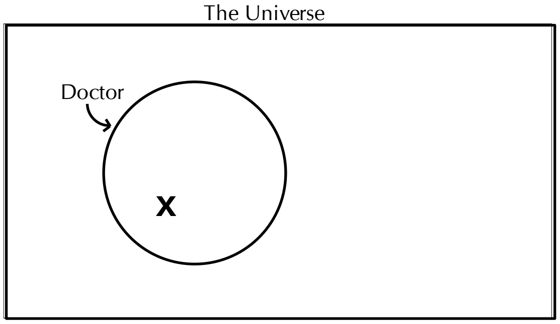
A predicate is a name of a ***class***, and a class is merely a collection of items which have something in common. What they have in common is whatever is identified by that very predicate. A class can be thought of as a conceptual container. If we could draw a very large circle on the ground and label it "Doctor", and if we could round up all doctors (if there are any) and place them in that circle, then the circle would represent the class or predicate Doctor. At the same time, the area outside of the circle would represent the *complement*, namely, the class or predicate Non-Doctor. It is called the complement (as opposed to the word "compliment" which means an expression of approval or flattery) because it *completes* something. In this case, the complement completes the universe of possibilities: once given the class Doctor, everything else in the universe must be in the class Non-Doctor. Here is a simple way to diagram the class Doctor and its complement Non-Doctor:



There are some rules for constructing and interpreting Venn diagrams, but they are easy to master.

Rule 1: *Existential commitments are indicated by an* **X** *in the appropriate area of the diagram.*

If there is something within the Doctor circle (i.e., if there is at least one thing which is a doctor), then we will represent that fact by putting an **X** in the circle. This will be a graphical representation of what is symbolized by (∃*x*)*Dx* (where *D* stands for the predicate "is a doctor"). And if we want to represent the claim that there is at least one non-doctor, then we will put an **X** in the Non-Doctor area, i.e., outside of the circle. In the diagram above, no claim at all is made about the existence or non-existence of non-doctors; the diagram says only that there are doctors. It does not say that everything is a doctor, nor does it say whether or not there are non-doctors. So we will observe this rule:

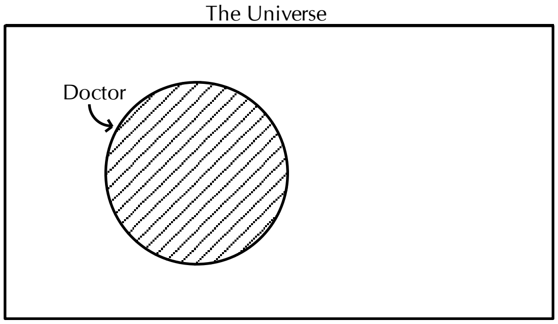


Rule 2: *A blank area on a diagram says nothing (except perhaps that we don't know whether or not there is anything in that area).*

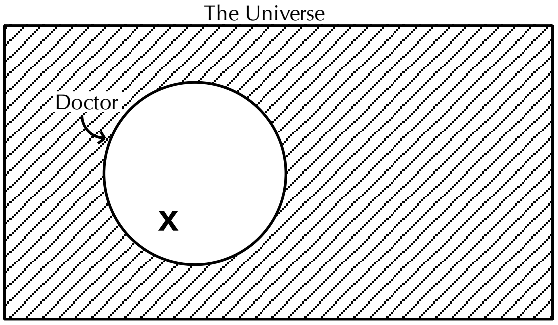
Blank areas are therefore areas of ignorance, ambiguity or indefiniteness. Blank areas are unfinished, and so they give us no information. If we want to diagram the statement that there are no doctors (i.e., −(∃*x*)*Dx*, or else (∀*x*)−*Dx*) then we cannot simply leave the Doctor circle blank. Rather, we will have to indicate in some positive way that nobody is home; we will have to make it impossible that there could be an **X** anywhere in that area. Here now is a third rule:

Rule 3: *An area which is claimed to be empty must be shaded (or hatched or otherwise filled in).*

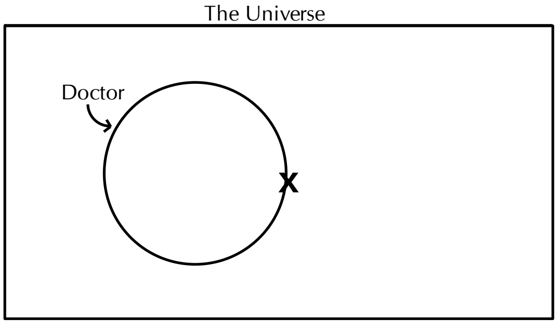
We fill in an area so that there is nowhere to put an **X** in that area. Here is how we could diagram the claim that there are no doctors:



And here is a diagram of the weird claim that at least one thing is a doctor, and nothing is a non-doctor:



Can you figure out what the following diagram claims?



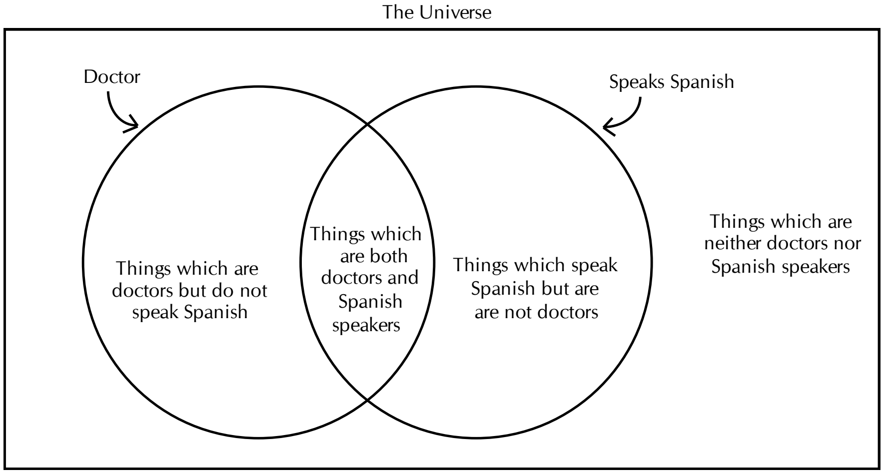
The **X** is sitting on the fence between Doctor and Non-Doctor, and therefore it is not definitely within one area rather than the other. However, this is *not* to be understood as the claim that there are both doctors and non-doctors. (Such a diagram must use two **X**s—one inside the Doctor circle, and one outside the Doctor circle.) Nor is this the self-contradictory claim that there is something which is both a doctor and not a doctor. Rather, it is simply the claim that there is something (the mere existence of the **X** tells us that). Just what this something is—whether it is or is not a doctor—is not given. The symbolic version of such a claim might be: $(\exists x)(Dx \vee -Dx)$. Thus we have still another rule:

Rule 4: *If you are going to place an* **X** *in the diagram, but there is not enough information given about exactly where to place that* **X***, then you must place it so that it straddles the lines between all areas where it might belong.*

A.2 Diagrams with Two Terms

Imagine that we have placed all the doctors in the Doctor circle, and we draw another large circle, label it "Speaks Spanish", and then go out in search of things to put in the Speaks Spanish circle. It ought to be no surprise that some doctors speak Spanish (or, equivalently, some Spanish speakers are doctors). So at least some of the things we want to put in the Speaks Spanish circle will already be in the Doctor circle. If we move one of those Spanish speaking doctors from the Doctor circle to the Speaks Spanish circle, will he/she cease to be a doctor? Presumably not! So we need some method of putting some things into the Doctor circle and into the Speaks Spanish circle at the same time. In order to be in two places at once, it would be convenient if those two places were the same place. But it would be inconvenient to have only one circle with two labels ("Doctor" and "Speaks Spanish"), because then we wouldn't know, for each item in the circle, whether it was a doctor-type thing or a Spanish speaking-type thing. (Perhaps such a circle would have to be named "Doctor *or* Spanish Speaker".)

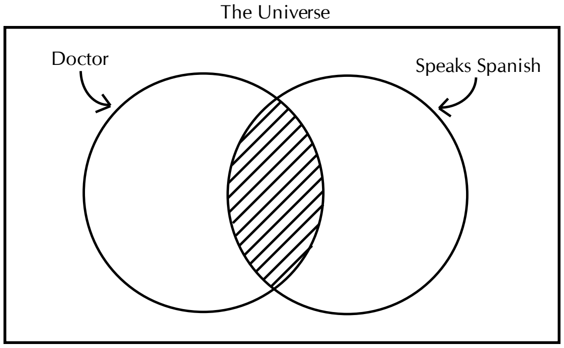
A simple solution presents itself: Just overlap the two circles. The result will be a conceptual division of items in the universe into four different classes:



Those four classes are (1) those things which are doctors but which do not speak Spanish, (2) those things which are doctors and which do speak Spanish, (3) those things which speak Spanish but which are not doctors, and finally (4) those things which are not doctors and also not Spanish speakers. We may symbolize the propositional functions for use in those four categories as follows:

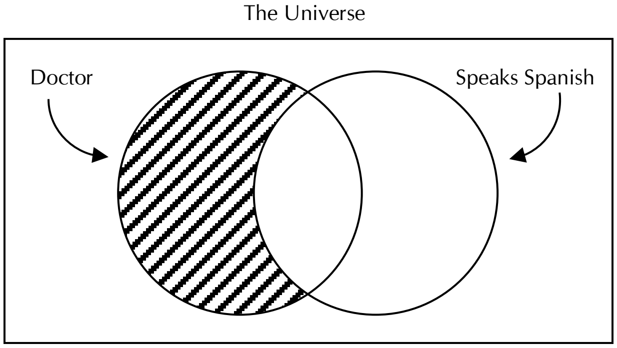
(1) $Dx \ \& \ -Sx$
(2) $Dx \ \& \ Sx$
(3) $-Dx \ \& \ Sx$
(4) $-Dx \ \& \ -Sx$

Our two-circled diagram will allow us to represent various propositions dealing with two predicates. For example, to diagram the statement that some doctors do not speak Spanish, i.e., $(\exists x)(Dx \,\&\, -Sx)$, we would merely place an **X** within that area of the Doctor circle which is not also within the Speaks Spanish circle. Or to diagram the claim that no doctors speak Spanish, we would follow Rule 3 and make sure that there could be no **X** within that intersecting area. Here is the diagram for $-(\exists x)(Dx \,\&\, Sx)$, which is logically equivalent to $(\forall x)(Dx \rightarrow -Sx)$:



How would you diagram the claim that no Spanish speakers are doctors? You would shade in the area of the Speaks Spanish circle which overlaps the Doctor circle. That is, it would be the very same diagram as above. Thus, the first claim, $-(\exists x)(Dx \;\&\; Sx)$ or equivalently $(\forall x)(Dx \rightarrow -Sx)$, must be logically equivalent to the second claim, which is $-(\exists x)(Sx \;\&\; Dx)$ or $(\forall x)(Sx \rightarrow -Dx)$. This can be verified by noting that the first member of either of those pairs can be derived from the first member of the other pair by means of the equivalence rule Commutation. The second member of each pair can be derived from the other by the equivalence rule Contraposition.

How would you diagram the claim that all doctors speak Spanish, i.e., $(\forall x)(Dx \rightarrow Sx)$? This may be puzzling at first, because we have at our disposal the symbol **X**, which we may put in some area of the diagram; and we may also shade in a certain area of the diagram. But **X** means "some", and shading means "none". How are we to represent "all"? The way to do it is to transform "all" into an equivalent "none". Look again at the symbolic version: $(\forall x)(Dx \rightarrow Sx)$. That sentence can also be translated a bit more literally as: "Given any x, if it is a doctor, then it speaks Spanish". In terms of our diagrams, that will mean: "If there is—or could be—any **X** in the Doctor area of the diagram, then it *must* fall into the Speaks Spanish area as well." And how do we insure that any **X** in the Doctor area would necessarily get into the Speaks Spanish area as well? Simple: We make sure that an **X** cannot go elsewhere. That is, if we shade in the area of Doctor which does *not* overlap the Speaks Spanish circle, then any **X** in the Doctor area must necessarily end up in the Speaks Spanish area. The diagram is given below. Notice that there is no **X** anywhere. Moreover, the entire Speaks Spanish circle is empty, and an empty area, according to Rule 2, says nothing about anything. Our diagram is supposed to represent that claim that all doctors speak Spanish, and yet the diagram neither affirms nor denies that there are any doctors to begin with! This may or may not be a problem with these kinds of diagram, depending on what metaphysical views you hold with regard to the nature of existence (and perhaps also the possibility of various kinds of existence). For example, you might want to be able to claim that all unicorns are white, without thereby having to claim that there really are any unicorns. In any case, the diagrams will agree nicely with our symbolic system, wherein $(\forall x)(Dx \rightarrow Sx) \vdash (\exists x)(Dx)$ is *not* a valid argument.



Exercise A.1

* Answers to starred problems are given in Appendix D.

Use Venn diagrams with two circles, *D* and *S*, to create diagrams for the following sentences.

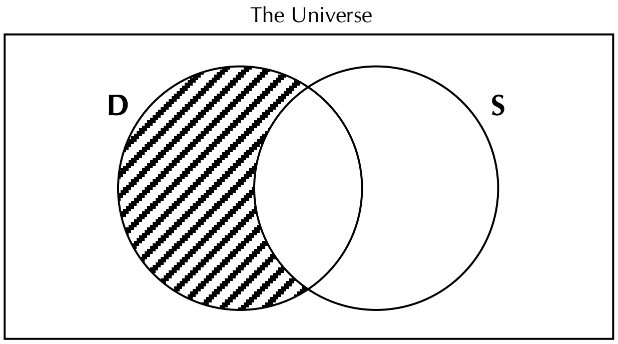
1. (∃x)(Dx & Sx) & (∃x)(Dx & –Sx)
2. (∀x)(Sx → –Dx)
* 3. –(∃x)Dx & (∃x)(Sx ∨ –Sx)
* 4. (∀x)(Dx ∨ Sx)
* 5. –(∃x)Sx
6. (∀x)Sx
7. (∃x)(Sx & –Dx)
8. (∀x)Dx & (∃x)Sx
9. (∀x)–Sx
10. –(∃x)(Dx ∨ –Sx)

A.3 Diagramming Arguments

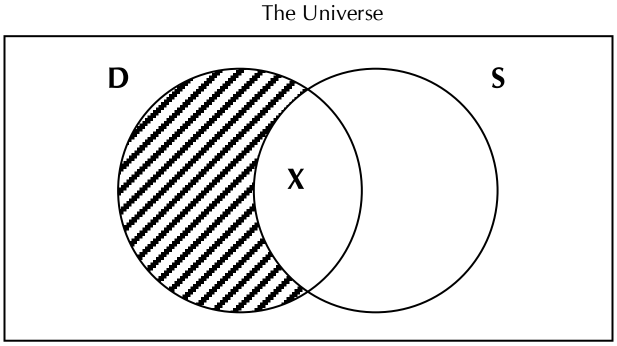
Venn diagrams can be used to test the validity of deductive arguments. The procedure is very simple: Draw as many different intersecting circles as there are different predicates in the sentences involved in the argument. Represent all the premises on the one diagram. (Do *not* include the conclusion.) If the diagram displays at least what would be necessary to represent the conclusion, then the argument is valid; otherwise, the argument is invalid. Here is an example of a very simple valid argument. The English sentences on the left are translated into symbolic notation on the right:

All doctors speak Spanish.	(∀x)(Dx → Sx)
There are doctors.	(∃x)Dx
Therefore, some Spanish speakers are doctors.	⊢ (∃x)(Sx & Dx)

Since there are only two different predicates in that argument (*D* and *S*), the Venn diagram method requires that we set up a universe with two classes, *D* and *S*, and then diagram the first premise:



Next we add the second premise on the same diagram. The claim that there are doctors requires that we place an **X** within the *D* circle, but we are not allowed to place it within a shaded region. (A shaded area, remember, means that no **X**s are allowed there.) So we have no choice but to place the **X** in the area where *D* and *S* overlap:

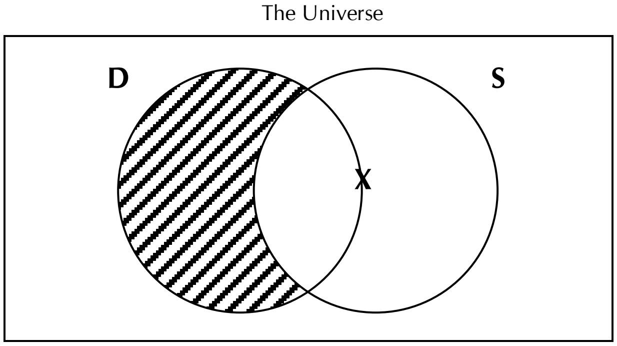


We have now diagrammed both premises, and we ask: What would be necessary in order to represent the conclusion? The conclusion claims that some Spanish Speakers are doctors, and the Venn diagram for such a claim would require an **X** in the area where *D* and *S* overlap. And, by golly, there is already an **X** there in our diagram; it was forced to be there because of the two premises. Hence, the argument is valid. Notice that the diagram contains more than what the conclusion requires: the shading of a portion of the Doctor circle is not a part of the conclusion itself. But that's OK. As long as what is necessary for the conclusion is unambiguously given on the diagram, it doesn't matter if other information is there as well.

Let's try another argument:

All doctors speak Spanish.	$(\forall x)(Dx \rightarrow Sx)$
There are Spanish speakers.	$(\exists x)Sx$
Hence, some Spanish speakers are doctors.	$\vdash (\exists x)(Sx \,\&\, Dx)$

The first premise is diagrammed just as it was for the previous argument. But the second premise is slightly different. To diagram the claim that there are Spanish speakers we need to place an **X** within the *S* circle. But there are two regions within the *S* circle, and so, following Rule 4, we must place the **X** ambiguously on the line between those two regions:

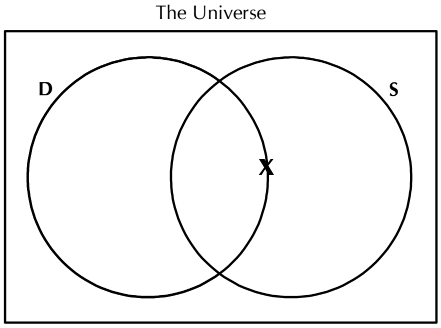


Now that both premises have been diagrammed, we look at the conclusion and ask what would be necessary to have it represented on the diagram as well. The conclusion, "Some Spanish speakers are doctors", would have to be represented by means of an **X** which would be in the *D* and *S* overlap area. Have the two premises automatically placed an **X** there? No. The **X** in the diagram *might* belong in the overlap region; but then again it might belong in the area of *S* which does not overlap *D*. Since the conclusion is not *definitely* given on that diagram, the argument is not valid.

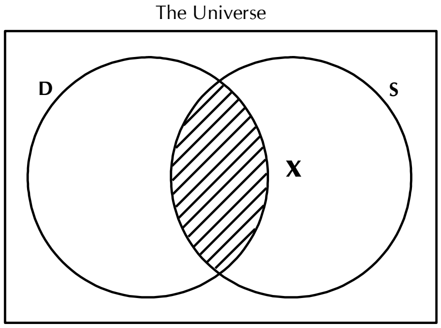
One more example:

There are Spanish speakers.	$(\exists x)Sx$
No doctors speak Spanish.	$(\forall x)(Dx \rightarrow -Sx)$
Hence, some Spanish speakers are not doctors.	$\vdash (\exists x)(Sx \ \& \ -Dx)$

The first premise requires an **X** in the *S* circle, but since there are two regions within the *S* circle, the **X** will have to sit on the fence:



The second premise requires shading. In particular, the claim that no *D* is *S* means that there cannot be any **X**s in the *D*-*S* overlap. Now notice that when we shade in that area, we automatically push that fence-sitting **X** off the fence:



The conclusion claims that there is an **X** in that part of the *S* circle which is not also within the *D* circle. And the two premises have automatically placed an **X** exactly there. Hence, the argument is valid.

Exercise A.2

* Answers to starred problems are given in Appendix D.

Use Venn diagrams to test the validity of the following arguments.

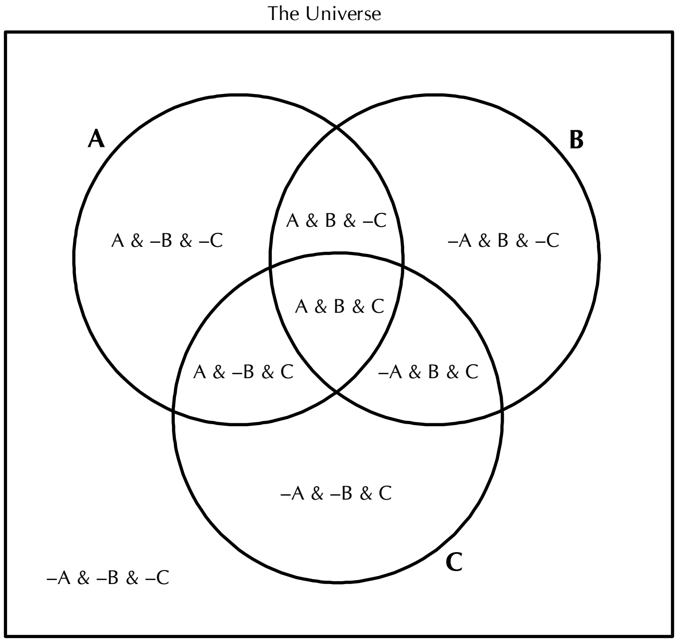
1. All doctors are Spanish speakers. There are no Spanish speakers. Hence, there are no doctors.
* 2. Some doctors do not speak Spanish. All Spanish speakers are doctors. Hence, some doctors speak Spanish.
3. There are no doctors. Hence, no doctors speak Spanish.

* 4. Everything is either a doctor or speaks Spanish. There are no doctors. Hence, everything is a Spanish speaker.

5. Some non-doctors do not speak Spanish. No doctors speak Spanish. Hence, some Spanish speakers are not doctors.

A.4 Diagrams with More than Two Classes

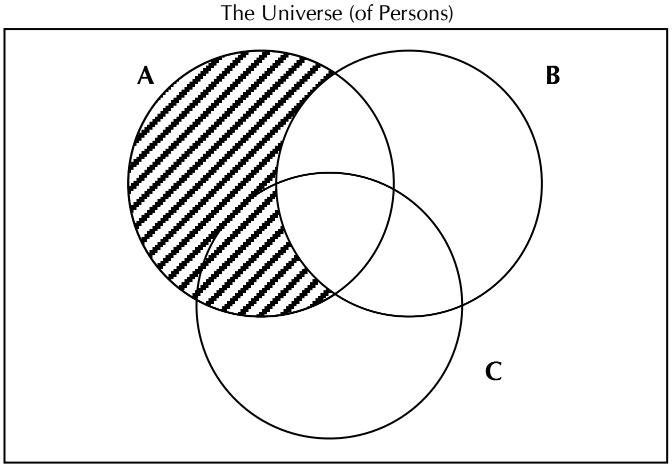
Venn diagrams with three circles are easily constructed. Three overlapping circles will divide the universe into eight different regions, as the diagram below illustrates. The principles involved with three classes are no different than with two classes, except that an extra level of care must be taken to make sure that **X**s and shadings go into the correct areas. With three different classes, the kinds of propositions and arguments which may be diagrammed increase substantially—that is, we are able to deal with more interesting material.



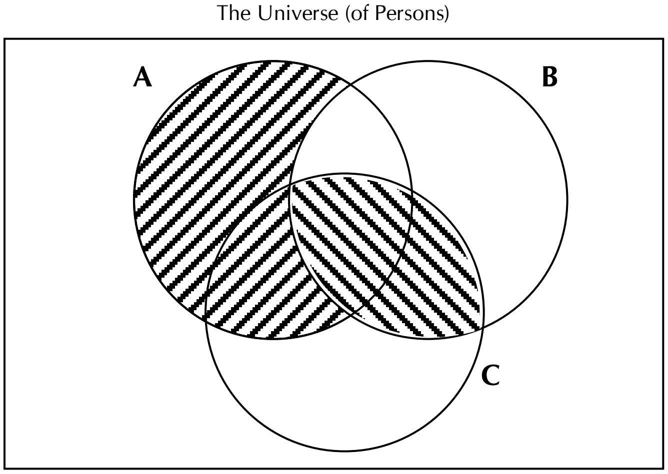
Let's test the validity of this argument:

All actors are boring.	$(\forall x)(Ax \rightarrow Bx)$
No composers are boring.	$(\forall x)(Cx \rightarrow\ -Bx)$
Hence, no composers are actors.	$\vdash\ (\forall x)(Cx \rightarrow\ -Ax)$

And for the sake of convenience, let us limit the universe to persons (rather than things). The first proposition is diagrammed by shading in all of the *A* circle which does not overlap the *B* circle. Since in this proposition we are dealing with only two of the three classes, we can neglect the *C* circle—just pretend it is not there:



The second premise claims that nothing is allowed in the *C*-*B* overlap area, and so we must shade it in:

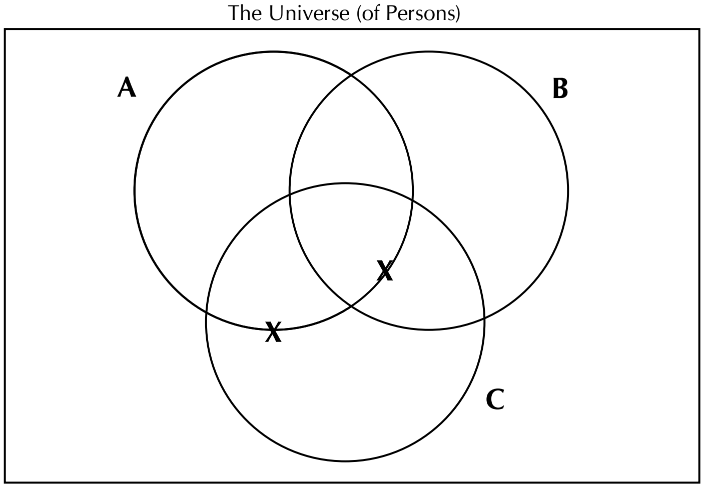


Now that all the premises have been diagrammed, the question is whether the conclusion has automatically been represented. And the answer is yes. The conclusion claims that no *C* is *A*, which means that the *C*-*A* overlap would have to be completely shaded. And indeed it has been shaded (partly by the action of the first premise, and partly by the action of the second premise). Hence, the agument is valid.

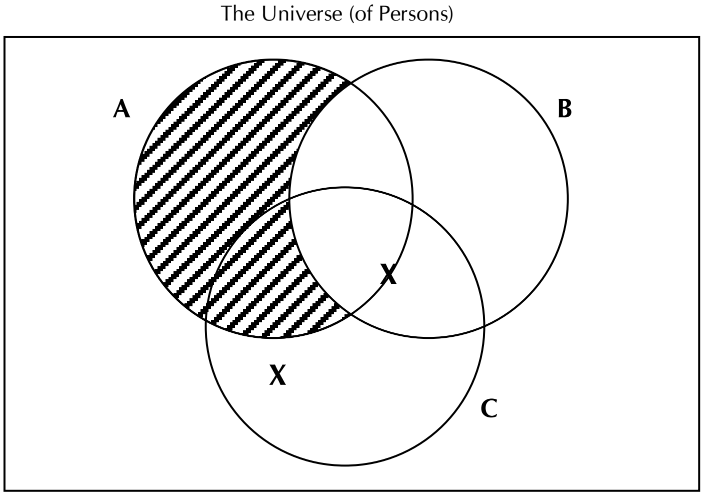
Let's try another:

Some composers are boring and some aren't.	$(\exists x)(Cx \,\&\, Bx) \,\&\, (\exists x)(Cx \,\&\, -Bx)$
All actors are boring.	$(\forall x)(Ax \to Bx)$
Hence, some composers are actors.	$\vdash (\exists x)(Cx \,\&\, Ax)$

The first premise actually makes two claims: first, there is an **X** in the *C*-*B* overlap, and second, there is an **X** in *C* which is not in the *C*-*B* overlap. Both of the **X**s will have to be fence-sitters with respect to the *A* circle, because we are not told whether those **X**s are or are not actors:



The second premise says that if there are going to be any **X**s in the *A* circle, then they must be forced to be in the *B* circle as well. This requires that we shade in the *A*-but-not-*B* area. As a result, one of the **X**s will get pushed off the fence:



Is the conclusion now represented? The conclusion would require an **X** in the *C*-*A* overlap. Well, there is an **X** in the *C*-*B* overlap, but it might or might not also be in the *A* circle. Since that **X** is sitting on the fence between *A* and not-*A*, it is not *definitely* in the *A* area, and hence the conclusion is not *definitely* represented. Hence the argument is not valid.

Exercise A.3

* Answers to starred exercises are given in Appendix D.

Use Venn diagrams to test the validity of these arguments.

* 1. All *A* is *B*. All *B* is *C*. Therefore, all *A* is *C*.
2. No *A* is *B*. No *B* is *C*. Therefore, no *A* is *C*.
* 3. Everything is either *A* or *B* or *C*. No *B* is *C*. There are *C*s. Hence, some *A* is *C*.

4. Some *A* is *B*. Some *C* is not *B*. Hence, some *C* is *A*.
5. All *A* is *B*. All *B* is *C*. Some *A* is *C*. Therefore, some *C* is *B*.
6. No *A* is *C*. Some *C* is *B*. So some *B* is not *A*.
7. Nothing is *A* unless it is *B*. Some things which are *B* happen to be *C*. Therefore, there is at least one thing which is both *A* and *C*.

* 8. Nothing which is both *A* and *B* is also *C*. Some things which are *C* are not *A*. If anything is *C*, then it must be either *A* or *B*. Hence, something is *B*.

9. Anything which is either *A* or *B* is also *C*. Some *A* is *B*. Hence, some *C* is *B*.

* 10. No non-*B* is *C*. No *A* is *B*. Hence, no *A* is *C*.

— B —

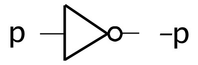
Appendix B: An Introduction to Digital Logic

With digital electronic circuits we may represent truth and falsity by a different—but still two-valued—system. Since we want to be concerned with some properties of electricity, we may choose any two incompatible but easily detectable properties. The two values might be a (relatively) high voltage and a (relatively) low voltage. How high “high” should be will depend upon the details of the circuit, but a very common value for “high” is 5 volts. A common value for “low” is 0 volts. We may say that 5 volts represents “on” or “true”, while 0 volts is “off” or “false”. (We could have chosen the opposite convention instead; it doesn’t matter, as long as we’re consistent in our use of whatever convention we choose.)

There are any number of ways of separating out 5 volts from 0 volts—that is, for detecting the existence of one or the other. The electronic devices used these days are transistors, usually of microscopic size fabricated on thin silicon wafers. We will represent such devices by means of fairly conventional symbols.

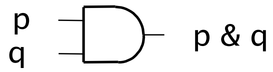
B.1 NOT, AND, OR and NAND

One combination of transistors is used to invert a signal: change 5 volts to 0 volts or vice versa. The exact configuration of such a system of transistors is not our concern. We need only represent such a device by this symbol:

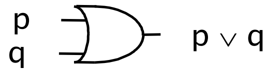


where a signal called *p* (5 volts or 0 volts) comes in from the left, and its inverted value goes out the right.

Still another combination of transistors performs the AND function. If both of two input signals, *p* and *q*, are 5 volts, then the output signal is 5 volts; otherwise the output is 0 volts:

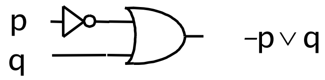


The (inclusive) OR is symbolized as

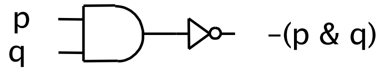


The output is 0 volts only when both inputs are 0 volts.

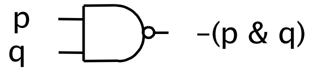
From our work in propositional logic, we know that these symbols are all we need. There is no conventional system of transistors which represents the conditional *p*→*q*, but we can represent it as a disjunction: −*p*∨*q*. So a digital logic circuit for a conditional might be this one:



A very popular device is the NAND (or NOT AND) switch, which corresponds to the symbolic formula −(*p*&*q*), and which, by DeMorgan's Law, is the same as −*p*∨−*q*:



Since NAND circuits are so popular among digital electronic designers (because, for one thing, all other functions can be built out of combinations of NANDs; see the Sheffer Stroke described in §3.13), its symbol is traditionally abbreviated to:



B.2 More Complex Functions

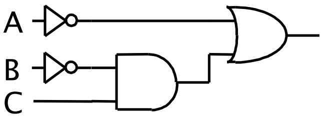
Given inverters, AND, OR and NAND devices (usually called "gates"), we can design more and more complex circuits which will be electronic examples of sentences in propositional logic. For example, the sentence

A → (−B & C)

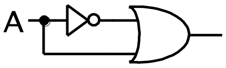
is first reinterpreted as a disjunction to get rid of the arrow:

−A ∨ (−B & C)

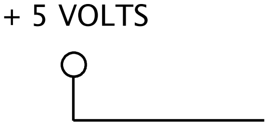
Now it may be implemented as



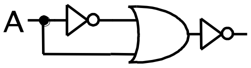
You can tell if a circuit represents a tautology if its output is always true, regardless of its input. We know, for example, that $A \lor -A$ is a tautology. An equivalent circuit might be:



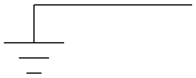
Of course, an even simpler circuit would just be a straight 5 volt signal:



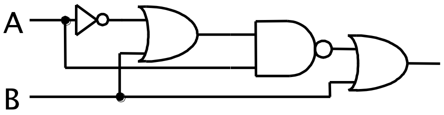
A contradiction is the opposite of (inversion of) a tautology:



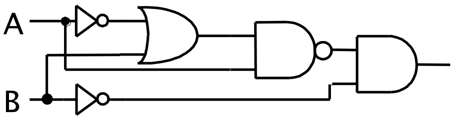
An even simpler circuit would be just a connection to ground (0 volts):



We could put together logic circuits for testing validity of arguments. Remember that if $p \vdash c$ is a valid argument, then $p \rightarrow c$ is a tautology. Here is a circuit for Modus Ponens (interpreting the conditional as a disjunction, and putting the two premises together as a conjunction):



We could, alternatively, test the validity of an argument electronically by means of a kind of Indirect Proof (Proof by Contradiction). Remember that if $p \vdash c$ is a valid argument, then p & $-c$ is inconsistent. Here is Modus Ponens again. The final output should always be 0 volts.



B.3 Disjunctive Normal Form

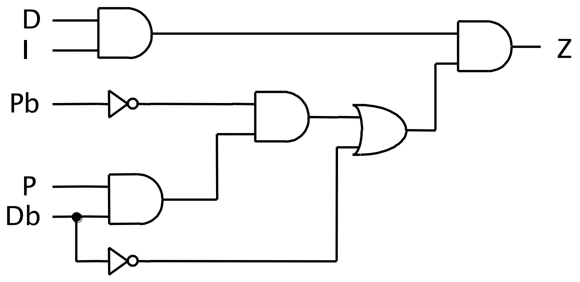
We may employ Disjunctive Normal Form to help construct an electronic circuit. Suppose, for example, we wish to describe a buzzer in an automobile which will warn the driver and/or passenger in case a seat belt is not buckled. Of course, we don't want the buzzer to go off if the passenger's seat belt is not buckled when there is no passenger. Nor do we care about the belts if there is no driver, nor when the ignition is off. And so on. We may draw up a chart (a truth table) which will describe the action of the buzzer under the various conditions. In the following table, *D* is true if there is someone in the driver's seat, *P* is true if someone is in the passenger's seat, *I* is true if the ignition is on, *Db* is true if the driver's seat belt is buckled, *Pb* is true if the passenger's seat belt is buckled, and *Z* is true if the buzzer sounds. Notice that the truth table really ought to consist of 32 rows, but the last 16 are cases where there is no driver, and so *Z* should be false.

D	P	I	Db	Pb	Z
T	T	T	T	T	F
T	T	T	T	F	T
T	T	T	F	T	T
T	T	T	F	F	T
T	T	F	T	T	F
T	T	F	T	F	F
T	T	F	F	T	F
T	T	F	F	F	F
T	F	T	T	T	F
T	F	T	T	F	F
T	F	T	F	T	T
T	F	T	F	F	T
T	F	F	T	T	F
T	F	F	T	F	F
T	F	F	F	T	F
T	F	F	F	F	F
		(all others)			F

The "buzzer sounds" sentence may be constructed directly from the table in Disjunctive Normal Form:

(D & P & I & Db & – Pb) ∨ (D & P & I & – Db & Pb)
∨ (D & P & I & – Db & – Pb) ∨ (D & – P & I & – Db & Pb)
∨ (D & – P & I & – Db & –Pb)

But if it is straightforwardly implemented electronically, it will be an overly complex circuit. We could simplify it in a number of ways (see section 4.7). Here is one possible result:



— C —
Appendix C: Prefix, Infix and Postfix Notation

C.1 Precedence

We have been writing sentences in this general form:

sub-sentence connective sub-sentence

But because one or both of the sub-sentences might themselves be complex, we had to resort to the use of parentheses in order to avoid ambiguity. For example, this sentence

A & B ∨ C

might be interpreted as a conjunction

A & (B ∨ C)

or as a disjunction

(A & B) ∨ C

Some uses of parentheses may be avoided if we were to establish a ***precedence*** among the various connectives such that connectives with higher precedence are to be interpreted first, and connectives with lower precedence next. Suppose, for example, the precedence order (from high to low) were

"–", "&", "∨", "→". Then the sentence

A & B ∨ C

would be taken as a *disjunction*, because the "&" would be understood as operating first, forming a conjunction of *A* and *B*, and the "∨" would be understood as operating on the result.

This is quite similar to multiplication and addition of numbers, where we have learned that multiplication takes precedence over addition. For example,

$$3 \times 2 + 1$$

is 7 (whereas if addition took precedence over multiplication, the result would be 9).

But even given a precedence order, parentheses cannot be avoided entirely, for sometimes we do not want the interpretation which precedence relations provide. Suppose, for example, we wish to add 2 to 1 and *then* multiply by 3. We have to use parentheses to override the usual precedence:

$$3 \times (2 + 1)$$

Similarly, if "&" has higher precedence than "∨", we still need parentheses if we wish to write a conjunction, where one of the conjuncts is a disjunction:

A & (B ∨ C)

C.2 Parenthesis-Free Notation

There are ways to express both arithmetic and logical operations without requiring parentheses. Our usual rules require that an operator be written between the ***operands*** (the sentences it operates on). This is convenient because it usually represents the order in which we express these elements in English. Let us call this ***operator infix notation***, or simply ***infix notation***. But we can use instead either ***operator prefix*** or ***operator postfix*** notation, where operators are written immediately before (for prefix) or immediately after (for postfix) their operands. Neither prefix nor postfix notation requires the use of parentheses.

Prefix Notation

In prefix notation* we write the operator followed immediately by its operand(s). A conjunction of *A* and *B*, for example, will be written as

&AB

instead of

A & B

A disjunction is rendered

---

* Also sometimes called **Polish notation** in honor of the Polish logician Jan Łukasiewicz who first proposed it. Postfix notation is sometimes called **reverse Polish notation**. Occasionally the phrases "Polish prefix" and "Polish postfix" are used.

∨AB

A conjunction whose second conjunct is a disjunction could be

&A∨BC

Notice how prefix notation works: In the example above, the first symbol tells us that the sentence is a conjunction, and so we merely mark off the rest of the sentence into two elements, each representing—in prefix form—one of the conjuncts. The first one happens to be *A*, which therefore represents the first conjunct. The second element begins with "∨", which tells us that it is a disjunction (in prefix form), and we therefore pick out the next two available elements for use as the two disjuncts. Thus the sentence would translate to infix notation as:

A & (B ∨ C)

Similarly, the sentence

&∨ABC

is a prefix version of the infix sentence

(A ∨ B) & C

The denial sign is a unary operator, that is, it takes only one operand. In prefix notation (just as in infix), it comes immediately prior to the sentence it acts on. For example, the infix sentence

A ∨ –(B & C)

is rendered in prefix form as

∨A–&BC

But notice that parentheses are never necessary.

Prefix notion corresponds to the way we form expressions in predicate notation. For examples, *Pa* says that the individual named *a* has the property *P*; and *Rab* says that *a* and *b* stand in a certain relation, *R*. A common way of expressing functions in many computer programming languages makes use of prefix notation: *sqrt*(*x*) might be a function to produce the square root of whatever value is given to it (instantiating the *x*); *delete*(*x*, *y*) might be a routine to delete an occurrence of the number or character *x* from the collection of numbers or characters *y*; *print*(*c*, *x*, *y*) might print the character *c* at screen coordinates *x*, *y*; and so on.

Postfix Notation

Prefix notation substituted position for parentheses. Postfix notation does the same, except that the operators now come *after* their operands. A conjunction, for example, is rendered in postfix as

AB&

If a sentence is to be denied, the denial sign comes after it:

A–

The infix sentence

A ∨ – (B & C)

is written in postfix form as

ABC& – ∨

Here are some sentences, written in prefix, infix and postfix for comparison:

Prefix	Infix	Postfix
P	P	P
– P	– P	P –
&PQ	P & Q	PQ&
&P – Q	P & – Q	PQ – &
∨→PQ – &R – S	(P → Q) ∨ – (R & – S)	PQ→RS – & – ∨
&P – ∨Q→RS	P & – (Q ∨ (R → S))	PQRS→∨ – &

C.3 Translating Infix to Postfix

You might find infix notation to be somehow more "natural", but this may simply reflect the habits you have nurtured since childhood. In any case, both prefix and postfix are more efficient, since neither parentheses nor precedence rules are required. It is on account of this efficiency that computers often make use of prefix or postfix arrangements in preference to infix. Some programming languages require that expressions be written by the programmer in postfix form. For example, a formula which we would ordinarily write in infix form, such as

$$(A + B) * (C - D)$$

(where "∗" represents multiplication) might have to be entered as a postfix expression:

$$A\ B + C\ D - *$$

Other programming languages allow the expression to be entered in infix form (under the assumption that the programmer is more comfortable with that); the computer will then translate the expression into postfix form for its own uses (thus relieving the programmer of that burden).

The procedure for translating infix expressions to postfix is both interesting and not difficult to program. (Translating to prefix involves some extra complications, and so we will not discuss it.) You can do it pretty much all "in your head", but computers, alas, lack that sort of skill, and so we must more rigorously describe the necessary steps.

To simplify matters, we shall assume that the infix expression is fully parenthesized. (We could, with only moderate additions, handle the precedence of operators.) In addition, we are supposing that the computer is given a syntactically correct expression to translate. That is, we will not bother with the extra complications necessary to include instructions for what to do in case a garbled expression is given (even though handling errors is an extremely important part of any program).

Here is the translation procedure:

Step 1. Going from left to right, get the next symbol from the infix expression. (At the very beginning, the "next" symbol will be the first symbol.) If there are no more symbols, go to Step 6.

Step 2. If the symbol is a letter, tack it on to the *right* of the postfix expression. (At the very beginning, there will be no postfix expression yet, and so the first letter got from the infix expression will simply go to start the postfix expression.) Then go back to Step 1.

Step 3. If it is a "(", save it in a temporary location, and go back to Step 1.

Step 4. If it is a ")", remove the *most recent* symbol stored in the temporary location and add it to the right of the postfix expression. Keep doing this until the symbol removed from the temporary location is a "(", in which case throw both parentheses away and go back to Step 1.

Step 5. Since the symbol obtained from the infix expression is neither a letter, nor "(", nor ")", it must be an operator. Add this new symbol to those already collected in the temporary location, and go back to Step 1.

Step 6. At this point, the entire infix expression has been examined. If there are any symbols left in the temporary location, remove the most recent one and add it to the right of the postfix expression. Continue doing this until there are no more symbols in the temporary location.

Step 7. Stop. The postfix expression is now complete.

Let's see this translation process in action on an example sentence:

(P ∨ Q) & −R

In the table below, an arrow, "↑", will point to the particular symbol that we are dealing with, and each time we have dealt with a symbol, we will move the arrow to the right one position.

Infix expression	**Postfix expression**	**Temporary**	**Step(s) used**
$\underset{\uparrow}{(}P \vee Q) \,\&\, -R$		(	1, 3
$(\underset{\uparrow}{P} \vee Q) \,\&\, -R$	P	(	1, 2
$(P \underset{\uparrow}{\vee} Q) \,\&\, -R$	P	(∨	1, 5
$(P \vee \underset{\uparrow}{Q}) \,\&\, -R$	PQ	(∨	1, 2
$(P \vee Q\underset{\uparrow}{)} \,\&\, -R$	PQ∨		1, 4
$(P \vee Q) \,\underset{\uparrow}{\&}\, -R$	PQ∨	&	5
$(P \vee Q) \,\&\, \underset{\uparrow}{-}R$	PQ∨	& –	5
$(P \vee Q) \,\&\, -\underset{\uparrow}{R}$	PQ∨R	& –	2
$(P \vee Q) \,\&\, -R\ \underset{\uparrow}{}$	PQ∨R –	&	6
$(P \vee Q) \,\&\, -R\ \underset{\uparrow}{}$	PQ∨R – &		6

— D —

Appendix D: Answers to Exercises (Starred Problems)

Answers to Exercises in Chapter 1

Exercise 1.1

(1.1) 1. False
(1.1) 3. True
(1.1) 5. True
(1.1) 7. False
(1.1) 9. True
(1.1) 13. $-(C \vee B)$, or $-C \ \& \ -B$. False.
(1.1) 15. $-[C \ \& \ (B \vee -A)]$. False.

Exercise 1.2

(1.2) 1.

A	B	C	–C	(B & –C)	A ∨ (B & –C)
T	T	T	F	F	T
T	T	F	T	T	T
T	F	T	F	F	T
T	F	F	T	F	T
F	T	T	F	F	F
F	T	F	T	T	T
F	F	T	F	F	F
F	F	F	T	F	F

(1.2) 3.

A	B	C	A ∨ C	B & (A ∨ C)
T	T	T	T	T
T	T	F	T	T
T	F	T	T	F
T	F	F	T	F
F	T	T	T	T
F	T	F	F	F
F	F	T	T	F
F	F	F	F	F

(1.2) 5.

A	B	C	C & A	–(C & A)	B & –(C & A)	A ∨ [B & –(C & A)]
T	T	T	T	F	F	T
T	T	F	F	T	T	T
T	F	T	T	F	F	T
T	F	F	F	T	F	T
F	T	T	F	T	T	T
F	T	F	F	T	T	T
F	F	T	F	T	F	F
F	F	F	F	T	F	F

(1.2) 7.

A	B	C	–B	–B ∨ C	–A	–A ⊕ (–B ∨ C)
T	T	T	F	T	F	T
T	T	F	F	F	F	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	T	F
F	T	F	F	F	T	T
F	F	T	T	T	T	F
F	F	F	T	T	T	F

(1.2) 9.

A	B	A & B	–(A & B)	(A & B) ⊕ –(A & B)
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Exercise 1.3

You will have to present both sentences of each pair on the same truth table, even though there is no connective ("&", "∨", etc.) which connects them. Then simply notice that the column of Ts and Fs under the one sentence is *exactly the same* as the column under the other sentence. (By the way, when this happens, the two sentences are said to be *logically equivalent*, as we'll see in section 1.4.)

(1.3) 1.

A	B	−A	−B	−A & −B	−(−A & −B)	A ∨ B
T	T	F	F	F	**T**	**T**
T	F	F	T	F	**T**	**T**
F	T	T	F	F	**T**	**T**
F	F	T	T	T	**F**	**F**
					⇧	⇧

(1.3) 3.

A	−A	− −A
T	F	**T**
F	T	**F**
⇧		⇧

(1.3) 5.

A	B	C	B & C	A ∨ (B & C)	A ∨ B	A ∨ C	(A ∨ B) & (A ∨ C)
T	T	T	T	**T**	T	T	**T**
T	T	F	F	**T**	T	T	**T**
T	F	T	F	**T**	T	T	**T**
T	F	F	F	**T**	T	T	**T**
F	T	T	T	**T**	T	T	**T**
F	T	F	F	**F**	T	F	**F**
F	F	T	F	**F**	F	T	**F**
F	F	F	F	**F**	F	F	**F**
				⇧			⇧

(1.3) 7.

A	A & A
T	**T**
F	**F**
⇧	⇧

(1.3) 9.

A	B	C	−A	−B	−A∨−B	−(−A∨−B)	−(−A∨−B)∨C	A∨C	B∨C	(A∨C)&(B∨C)
T	T	T	F	F	F	T	**T**	T	T	**T**
T	T	F	F	F	F	T	**T**	T	T	**T**
T	F	T	F	T	T	F	**T**	T	T	**T**
T	F	F	F	T	T	F	**F**	T	F	**F**
F	T	T	T	F	T	F	**T**	T	T	**T**
F	T	F	T	F	T	F	**F**	F	T	**F**
F	F	T	T	T	T	F	**T**	T	T	**T**
F	F	F	T	T	T	F	**F**	F	F	**F**
							⇧			⇧

Exercise 1.4

(1.4) 1b.

A	−A	A & −A
T	F	F
F	T	F

Inconsistent, because there is not at least one T in the final column.

(1.4) 1d.

A	B	−B	A∨−B	(A∨−B)&B
T	T	F	T	**T**
T	F	T	T	F
F	T	F	F	F
F	F	T	T	F

⇦ Consistent

(1.4) 2a.

①			③	②
A	B	−A	−B	B ∨ −A
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Inconsistent set, because there is no row which has T under each of the three sentences.

(1.4) 2c.

①	③	②	④
A	B	A ∨ −A	B ∨ −B
T	T	T	T
T	F	T	T
F	T	T	T
F	F	T	T

Consistent set, because there is at least one row which has T under each of the four sentences.

(1.4) 3a.

A
T
F

Contingent, because the sentence has at least one T and at least one F.

(1.4) 3c.

A	B	A&B	−A	−B	−A∨−B	(A&B)∨(−A∨−B)	−[(A&B)∨(−A∨−B)]
T	T	T	F	F	F	T	F
T	F	F	F	T	T	T	F
F	T	F	T	F	T	T	F
F	F	F	T	T	T	T	F
							⇧

Contradictory (i.e., inconsistent).

(1.4) 4b.

A	C	−A	C&−A	−C	−C∨A	−(−C∨A)
T	T	F	**F**	F	T	**F**
T	F	F	**F**	T	T	**F**
F	T	T	**T**	F	F	**T**
F	F	T	**F**	T	T	**F**
			⇧			⇧

Yes; the two sentences have identical columns in the truth table.

(1.4) 4e.

A	B	–B	A & –B	B & –B
T	T	F	**F**	**F**
T	F	T	**T**	**F**
F	T	F	**F**	**F**
F	F	T	**F**	**F**
			⇧	⇧

No; the columns are not identical.

(1.4) 5. Yes. All tautologies have nothing but Ts in the final columns of their truth tables. Sentences which have the same truth tables are logically equivalent. So all tautologies, having identical truth tables, are logically equivalent.

(1.4) 7. Call the tautology $\mathcal{T}$, and call the contradiction $\mathcal{F}$. The truth table for their conjunction would have this general form:

$\mathcal{T}$	$\mathcal{F}$	$\mathcal{T}$ & $\mathcal{F}$
T	F	F
T	F	F
⋮	⋮	⋮
T	F	F

So the conjunction is a contradiction, which is one kind of logically determinate sentence. (The other kind is tautology.)

(1.4) 9. No. A tautology has all Ts in its truth table column; a contradiction has all Fs. There is therefore no row in the truth table with a T under both. Hence, they are inconsistent with each other.

Exercise 1.5

(1.5) 3. Being a mother is a sufficient condition for being a relative. That is, if you are a mother, then you are automatically a relative of some sort. Being a relative is a necessary condition for being a mother. But of course it is not sufficient, because some relatives are not mothers.

Exercise 1.6

(1.6) 1.

S	P	Q	–S	P&Q	–S↔(P&Q)	[–S↔(P&Q)]∨S
T	T	T	F	T	F	T
T	T	F	F	F	T	T
T	F	T	F	F	T	T
T	F	F	F	F	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	F
F	F	T	T	F	F	F
F	F	F	T	F	F	F

Since there are both Ts and Fs in the sentence's column, the sentence is contingent.

(1.6) 3.

A	B	A & B	–A	–B	–A ∨ –B	–(–A∨–B)	(A & B) ↔ –(–A∨–B)
T	T	T	F	F	F	T	T
T	F	F	F	T	T	F	T
F	T	F	T	F	T	F	T
F	F	F	T	T	T	F	T

The sentence is always true, so it is a tautology.

(1.6) 8.

R	R ↔ R	R ↔ (R ↔ R)
T	T	T
F	T	F

Since there is at least one T and one F in the sentence's column, the sentence is contingent.

Exercise 1.7

(1.7) 1.

A	B	A ∨ B	B & (A ∨ B)	
T	T	T	T	⇨ (A & B)
T	F	T	F	
F	T	T	T	⇨ (–A & B)
F	F	F	F	⇩

Disjunctive Normal Form: (A & B) ∨ (–A & B)

(Note: The final column in the truth table is identical with the column under *B*. But *B* is, strictly speaking, not in full Disjunctive Normal Form.)

(1.7) 3.

A	B	A ∨ B	–(A ∨ B)	–A	B ↔ –A	–(A ∨ B) → (B ↔ –A)	
T	T	T	F	F	F	T	⇨ A & B
T	F	T	F	F	T	T	⇨ A & –B
F	T	T	F	T	T	T	⇨ –A & B
F	F	F	T	T	F	F	⇩

Disjunctive Normal Form: (A & B) ∨ (A & –B) ∨ (–A & B)

(1.7) 6. (D & E) ∨ (–D & E) ∨ (–D & –E)

(1.7) 10. (A & –Y) ∨ (–A & –Y)

(1.7) 11.

A	B	A & B	B ∨ (A & B)	
T	T	T	T	
T	F	F	F	⇨ –A ∨ B
F	T	F	T	
F	F	F	F	⇨ A ∨ B

⇩

Conjunctive Normal Form: (–A ∨ B) & (A ∨ B)

(1.7) 13. (–A ∨ –W) & (A ∨ W)

(1.7) 15. –A ∨ –B

Answers to Exercises in Chapter 2

Exercise 2.1

(2.1) 1. Conclusion: "A rate increase would be acceptable to the membership." Premises: "Pat's report." (Presumably Pat's report has some relevant information.) "The treasurer's analysis." (Presumably the analysis also has relevant information.) This is evidently an inductive argument, since it does not claim absolute certainty for the conclusion: the committee "feels" that a rate increase would be acceptable.

(2.1) 3. Conclusion: "Today the mouse will play." Premises: "When the cat's away, the mouse will play." "And today the cat is away." This is a deductive argument. Note that the question "Why?" is a lead-in for presenting the premises—that is, the reasons, which is just another term for "premises".

(2.1) 5. Conclusion: "Not everything is determined." Premises: "If everything is determined, then there is no true free choice." "And if there is no free choice, then there is no place for praise and blame." "Praise and blame do indeed attach to people's actions." This is a deductive argument: it appears to try to establish its conclusion conclusively on the basis of its premises. Note that the word "because" is a pretty good indicator that one or more premises will follow. Note also that although we might be forced to accept the conclusion if we accepted the premises, one of the premises is extremely weak: "Praise and blame do indeed attach to people's actions." That is a key premise, and we might well ask for a good deal of discussion before we felt constrained to accept it.

(2.1) 10. Conclusion: "Johnson is a bit of a hypocrite." Premises: "He's all the time telling everyone about the evils of eating meat, junk food, fatty foods, fried foods, etc.", and "Yesterday I caught him eating a hamburger and fries at a local fast food restaurant." This is a deductive argument. There is a missing—but obvious—premise, namely, the definition of a hypocrite. Perhaps something like this would do: "A hypocrite is a person who advises one thing but does another."

Exercise 2.2

(2.2) 1.

		Concl			**Prem 1**	**Prem 2**	
A	B	C	A&B	$-C$	$(A\&B)\vee -C$	$-B$	
T	T	T	T	F	T	F	
T	T	F	T	T	T	F	
T	F	T	F	F	F	T	
T	F	**F**	F	T	**T**	**T**	⇦ Invalid, because there is
F	T	T	F	F	F	F	at least one way (in this
F	T	F	F	T	T	F	problem there are two ways)
F	F	T	F	F	F	T	to make the premises true
F	F	**F**	F	T	**T**	**T**	⇦ and the conclusion false.

(2.2) 5.

			Prem 1	**Prem 2**	**Prem 3**			**Concl.**
A	Q	F	–A	–Q	–F	–Q→F	A ∨ (–Q→F)	–[A∨(–Q→F)]
T	T	T	F	F	F	T	T	F
T	T	F	F	F	T	T	T	F
T	F	T	F	T	F	T	T	F
T	F	F	F	T	T	F	T	F
F	T	T	T	F	F	T	T	F
F	T	F	T	F	T	T	T	F
F	F	T	T	T	F	T	T	F
F	F	F	T	T	T	F	F	T

Valid, because there is no row with T under each premise and F under the conclusion.

(2.2) 7.

				Prem 1		**Prem 2**				**Concl**
H	L	M	C	H∨L	M∨C	L→(M∨C)	–H	–M	–H&–M	(–H&–M)→C
T	T	T	T	T	T	T	F	F	F	T
T	T	T	F	T	T	T	F	F	F	T
T	T	F	T	T	T	T	F	T	F	T
T	T	F	F	T	F	F	F	T	F	T
T	F	T	T	T	T	T	F	F	F	T
T	F	T	F	T	T	T	F	F	F	T
T	F	F	T	T	T	T	F	T	F	T
T	F	F	F	T	F	T	F	T	F	T
F	T	T	T	T	T	T	T	F	F	T
F	T	T	F	T	T	T	T	F	F	T
F	T	F	T	T	T	T	T	T	T	T
F	T	F	F	T	F	F	T	T	T	F
F	F	T	T	F	T	T	T	F	F	T
F	F	T	F	F	T	T	T	F	F	T
F	F	F	T	F	T	T	T	T	T	T
F	F	F	F	F	F	T	T	T	T	F

Valid, because there is no way to make all the premises T and the conclusion F at the same time.

(2.2) 9.

	Concl		**Prem**	
E	A	E → E	E → (E → E)	
T	T	T	T	
T	**F**	T	**T**	⇦
F	T	T	T	
F	**F**	T	**T**	⇦

Invalid, because there is at least one way (in this problem there are two ways) to make the premise true and the conclusion false.

(2.2) 14. A, B ∨ C ⊢ A & C

Prem 1			**Prem 2**	**Concl**		
A	B	C	B∨C	A&C		
T	T	T	T	T		
T	T	F	**T**	**F**	⇦	Invalid
T	F	T	T	T		
T	F	F	F	F		
F	T	T	T	F		
F	T	F	T	F		
F	F	T	T	F		
F	F	F	F	F		

(2.2) 15. (L & M) → –R, R ⊢ –L

		Prem 2			**Prem 1**	**Concl**		
L	M	R	L&M	–R	(L&M) → –R	–L		
T	T	T	T	F	F	F		
T	T	F	T	T	T	F		
T	F	**T**	F	F	**T**	**F**	⇦	Invalid
T	F	F	F	T	T	F		
F	T	T	F	F	T	T		
F	T	F	F	T	T	T		
F	F	T	F	F	T	T		
F	F	F	F	T	T	T		

(2.2) 18. L ∨ M ⊢ –(–L & –M)

L	M	L ∨ M	–L	–M	–L & –M	–(–L & –M)
T	T	T	F	F	F	T
T	F	T	F	T	F	T
F	T	T	T	F	F	T
F	F	F	T	T	T	F

Valid.

(2.2) 19. T → N, –N ⊢ –T

		Prem 1	**Prem 2**	**Concl**
T	N	T→N	–N	–T
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Hey! This argument, which says of itself that it is *not* valid, *is* valid!

Exercise 2.3

(2.3) 1.
T F F F F
(A & B) ∨ −C, −B ⊢ C
true true false
Invalid.

(2.3) 5.
F F F F F F
−A, −Q, −F ⊢ −[A ∨ (−Q → F)]
true true true true
Valid.

(2.3) 7.
F F F F F F F F
H ∨ L, L → (M ∨ C) ⊢ (−H & −M) → C
false true false
Valid.

(2.3) 9.
F F
E → (E → E) ⊢ A
true false

(2.3) 14.
T T F T F
A, B ∨ C ⊢ A & C
true true false
Invalid.

(2.3) 15.
T F T T T
(L & M) → −R, R ⊢ −L
true true false
Invalid.

(2.3) 18.
F F F F
L ∨ M ⊢ −(−L & −M)
false false
If you make the conclusion false, then the premise turns out to be false. There is no way to make the premise true and the conclusion false, so the argument is valid.

(2.3) 19.
T F F T
T → N, −N ⊢ −T
false true false
Valid.

Exercise 2.4

(2.4) 3. Given a valid argument, there is no row in its truth table which has T under each premise and F under the conclusion. Suppose now another premise is added to the original set of premises. The new premise must be either T or F. But whichever it is, there *still* won't be a situation where the argument will have *all* true premises and a false conclusion.

(2.4) 5. Another way to phrase this metalogical truth is: If a given argument is valid, and if its premise is a tautology, then its conclusion *must* also be a tautology. And that's easy to show: Suppose the conclusion *weren't* a tautology. Then it would have one or more Fs in its column of the truth table. We are given that the premise is a tautology, so we know that its column has only Ts, and one of them would line up with an F in the conclusion, if the conclusion did indeed have any Fs in it:

premise	conclusion	
T	.	
T	.	
T	F	⇦
T	.	
T	.	

But then we would have an invalid argument. But since in this problem we are supposed to begin with the assumption that the argument is valid, we know that an F cannot occur in the conclusion after all. And a sentence whose truth table has no Fs in it is by definition a tautology.

(2.4) 9. We are given that *s*⊢*p* is a valid argument (where *s* is some set of premises, and *p* is the conclusion). *s* and *p* might be simple sentences, or they might be very complex; we don't know. But if you were to construct a truth table for them you would find some or all of the following possibilities *except for row 2* (which would show up only when an argument is invalid):

	s	p	−p	
1.	T	T	F	
2.	**T**	**F**	T	⇦
3.	F	T	F	
4.	F	F	T	

Since the argument is given as valid, this kind of row will not show up.

Since row 2 is ruled out, that also rules out the only row where both *s* and −*p* are true at the same time; in the three remaining rows, *s* and −*p* are *not* both true. Hence, by definition, *s* and −*p* constitute an inconsistent set.

(2.4) 11. We are given that *s* is some sentence or other which is consistent. Since it is consistent, then, by definition, there must be at least one T in its column in the truth table. We are told nothing about the conclusion, *p*, except that it follows validly from *s*. But that's enough information to deduce that *p*'s column in the truth table cannot have an F where *s*'s column has a T. (Because that would amount to the invalidity of *s*⊢*p*.) Since we know that *s* has at least one T, and we know that in that same row *p* must also have a T, then by definition *s* and *p* form a consistent set.

Answers to Exercises in Chapter 3

Exercise 3.2

(3.2) 3. Reduction to Absurdity.
(3.2) 6. Disjunctive Syllogism.
(3.2) 11. Weakening.
(3.2) 15. Disjunctive Syllogism.
(3.2) 19. Absorption.

Exercise 3.3

(3.3) 2. Material Equivalence.
(3.3) 5. Distributive Law.
(3.3) 7. DeMorgan's Law.
(3.3) 10. Distributive Law.
(3.3) 15. Exportation.

Exercise 3.4

(3.4) 1. Contraposition, Double Negation, Hypothetical Syllogism.
(3.4) 3. Weakening, Commutation, Modus Ponens, Association, Separation.
(3.4) 5. Distribution, Commutation, Separation, Idempotency, Association, Disjunctive Syllogism, Material Implication.
(3.4) 7. Material Equivalence, Separation, Separation, Modus Ponens.
(3.4) 9. Absorption, Hypothetical Syllogism, Separation, Modus Tollens, DeMorgan's.

Exercise 3.5

(3.5) 1.

1.	$A \to -B$	
2.	$C \to B$	$\vdash A \to -C$
3.	$-B \to -C$	2, Contraposition
4.	$A \to -C$	1,3, Hypothetical Syllogism

(3.5) 5.

1.	$[(M \& N) \& O] \to P$	
2.	$Q \to [(O \& M) \& N]$	$\vdash -Q \vee P$
3.	$[O \& (M \& N)] \to P$	1, Commutation
4.	$[(O \& M) \& N] \to P$	2, Association
5.	$Q \to P$	2,4, Hypothetical Syllogism
6.	$-Q \vee P$	5, Material Implication

(3.5)	8.	1.	A → L	
		2.	−A → U	
		3.	L → C	
		4.	(U ∨ C) → B	⊢ B
		5.	−A ∨ L	1, Material Implication
		6.	U ∨ C	2,3,5, Dilemma
		7.	B	4,7, Modus Ponens

(3.5)	10.	1.	−A ∨ (M & E)	
		2.	−E	
		3.	A ∨ (C → −R)	⊢ −(C & R)
		4.	(−A ∨ M) & (−A ∨ E)	1, Distribution
		5.	−A ∨ E	4, Separation
		6.	−A	5,2, Disjunctive Syllogism
		7.	C → −R	3,6, Disjunctive Syllogism
		8.	−C ∨ −R	7, Material Implication
		9.	−(C & R)	8, DeMorgan's Law

Exercise 3.6

(3.6)	1.	1.	(E ∨ S) → (A & H)	
		2.	(A ∨ H) → I	
		3.	E	⊢ I
		4.	E ∨ S	3, Weakening
		5.	A & H	1,4, Mod. Pon.
		6.	A	5, Separation
		7.	A ∨ H	6, Weakening
		8.	I	2,7, Mod. Pon.

(3.6)	3.	1.	F → W	⊢ (F & S) → W
		2.	−F ∨ W	1, Mat. Imp.
		3.	(−F ∨ W) ∨ −S	2, Weakening
		4.	−F ∨ (W ∨ −S)	3, Association
		5.	−F ∨ (−S ∨ W)	4, Commutation
		6.	(−F ∨ −S) ∨ W	5, Association
		7.	−(F & S) ∨ W	6, DeMorgan's
		8.	(F & S) → W	7, Mat. Imp.

(3.6)	7.	1.	C → (E & P)	
		2.	P → U	
		3.	C	⊢ U
		4.	E & P	1,3, Mod. Pon.
		5.	P	4, Separation
		6.	U	2,5, Mod. Pon.

(3.6) 9.

1.	B → −F	
2.	C → F	⊢ C → −B
3.	F → −B	1, Contraposition
4.	C → −B	2,3, Hyp. Syll.

Exercise 3.7

(3.7) 2.

1.	G	⊢ G → [G → (G → G)]
2.	G ∨ −G	1, Weakening
3.	−G ∨ G	2, Commutation
4.	G → G	3, Mat. Imp.
5.	(G → G) ∨ −G	4, Weakening
6.	−G ∨ (G → G)	5, Commutation
7.	G → (G → G)	6, Mat. Imp.
8.	[G → (G → G)] ∨ −G	7, Weakening
9.	−G ∨ [G → (G → G)]	8, Commutation
10.	G → [G → (G → G)]	9, Mat. Imp.

(3.7) 3.

1.	[(Y & Z) → A] & [(Y & B) → C]	
2.	(B ∨ Z) & Y	⊢ A ∨ C
3.	(Y & Z) → A	1, Separation
4.	(Y & B) → C	1, Separation
5.	Y & (B ∨ Z)	2, Commutation
6.	Y & (Z ∨ B)	5, Commutation
7.	(Y & Z) ∨ (Y & B)	6, Distribution
8.	A ∨ C	3,4,7, Dilemma

(3.7) 7.

1.	−(S & M) → O	
2.	−M	⊢ O
3.	−M ∨ −S	2, Weakening
4.	−(M & S)	3, DeMorgan's
5.	−(S & M)	4, Commutation
6.	O	1,5, Modus Ponens

(3.7) 11.

1.	−M	
2.	B	
3.	R → E	
4.	B → R	
5.	(E & A) → M	⊢ −A
6.	−(E & A)	5,1, Modus Tollens
7.	−E ∨ −A	6, DeMorgan's
8.	R	4,2, Modus Ponens
9.	E	3,8, Modus Ponens
10.	−A	7,9, Disjunctive Syllogism

(3.7) 16.

1.	$C \rightarrow (R \rightarrow A)$	
2.	$-A$	
3.	R	$\vdash -C$
4.	$-A \ \& \ R$	2,3, Conjunction
5.	$-(A \vee -R)$	4, DeMorgan's
6.	$-(-A \rightarrow -R)$	5, Material Implication
7.	$-(R \rightarrow A)$	6, Contraposition
8.	$-C$	1,7, Modus Tollens

(3.7) 17.

1.	$-D \rightarrow -C$	
2.	$-B \rightarrow -A$	
3.	$B \rightarrow C$	
4.	$D \rightarrow E$	$\vdash -(A \ \& \ -E)$
5.	$C \rightarrow D$	1, Contra.
6.	$B \rightarrow D$	3,5, Hyp. Syll.
7.	$A \rightarrow B$	2, Contra.
8.	$A \rightarrow D$	7,6, Hyp. Syll.
9.	$A \rightarrow E$	8,4, Hyp. Syll.
10.	$-A \vee E$	9, Mat. Imp.
11.	$-(A \ \& \ -E)$	10, DeMorgan's

Exercise 3.8

(3.8) 1.

1.	$p \rightarrow q$	
2.	$-q$	$\vdash -p$
3.	$-p \vee q$	1, Mat. Imp.
4.	$-p$	3,2, Dis. Syll.

(3.8) 3.

1.	$(p \ \& \ q) \rightarrow r$	$\vdash p \rightarrow (q \rightarrow r)$
2.	$-(p \ \& \ q) \vee r$	1, M.I.
3.	$(-p \vee -q) \vee r$	2, DeMorgan's
4.	$-p \vee (-q \vee r)$	3, Association
5.	$-p \vee (q \rightarrow r)$	4, M.I.
6.	$p \rightarrow (q \rightarrow r)$	5, M.I.

Notice that Exportation is an Equivalence, and the above proof proves only one half of the equivalence. (Recall that if any two sentences are logically equivalent to each other, then each validly implies the other. That is, if $r \equiv s$, then both $r \vdash s$ and $s \vdash r$ are valid.) Now for the other half of the proof:

1.	$p \rightarrow (q \rightarrow r)$	$\vdash (p \ \& \ q) \rightarrow r$
2.	$p \rightarrow (-q \vee r)$	1, M.I.
3.	$-p \vee (-q \vee r)$	2, M.I.
4.	$(-p \vee -q) \vee r$	3, Association
5.	$-(p \ \& \ q) \vee r$	4, DeMorgan's
6.	$(p \ \& \ q) \rightarrow r$	5, M.I.

Exercise 3.9

```
(3.9)  1.    1. | A → (B ∨ C)
             2. | B → D
             3. | C → D                     ⊢ A → D
          ------|-----------------------
             4. |         | A                       Cond. Assump.
                |         |-------------------
             5. |         | B ∨ C                   1,4, M.P.
             6. |         | D ∨ D                   2,3,5, Dilemma
             7. |         | D                       6, Idempotency
             8. | A → D                             4 → 7, C.P.
```

```
(3.9)  5.    1. | A → (−B ∨ C)
             2. | D → (C ∨ E)
             3. | −C                                ⊢ B → [−E → (−A & −D)]
          ------|---------------------------------
             4. |         | B                                   Cond. Assump.
                |         |-------------------------------
             5. |         |          | −E                       Cond. Assump.
                |         |          |--------------------
             6. |         |          | −C & −E                  3,5, Conj.
             7. |         |          | −(C ∨ E)                 6, DeM.
             8. |         |          | −D                       2,7, M.T.
             9. |         |          | A → (B → C)              1, M.I.
            10. |         |          | (A & B) → C              9, Exp.
            11. |         |          | −(A & B)                 10,3, M.T.
            12. |         |          | −A ∨ −B                  11, DeM.
            13. |         |          | −A                       12,4, D.S
            14. |         |          | −A & −D                  13,8, Conj.
            15. |         | −E → (−A & −D)                      5 → 14, C.P.
            16. | B → [−E → (−A & −D)]                          4 → 15, C.P.
```

```
(3.9)  9.    1. | C → R
             2. | L → Y
             3. | −L → C            ⊢ Y ∨ R
          ------|-----------------
             4. |        | −Y        Cond. Assump.
                |        |--------
             5. |        | −L        2,4, M.T.
             6. |        | C         3,5, M.P.
             7. |        | R         1,6, M.P.
             8. | −Y → R             4 → 7, C.P.
             9. | Y ∨ R              8, Mat. Imp.
```

```
(3.9) 11.    1. | C                 ⊢ A → A
          ------|-----------
             2. |       | A           Conditional Assumption
                |       |---------
             3. |       | A           2, Repetition
             4. | A → A               2 → 3, C.P.
```

(3.9) 13.

1.	C → M	
2.	−M ∨ (X & Y)	
3.	−R → −Y	⊢ C → R
4.	C	C.A.
5.	M	1,4, Mod. Pon.
6.	X & Y	2,5, Dis. Syll.
7.	Y	6, Sep.
8.	R	3,7, Mod. Tol.
9.	C → R	4 → 8, C.P.

(3.9) 15.

1.	(D & E) ∨ −(H ↔ O)	
2.	D → (E → P)	⊢ (H ↔ O) → P
3.	H ↔ O	C.A.
4.	D & E	1,3, Dis. Syll.
5.	(D & E) → P	2, Exp.
6.	P	5,4, M.P.
7.	(H ↔ O) → P	3 → 6, C.P.

Exercise 3.10

(3.10) 1.

1.	A → B	
2.	C → D	
3.	(B ∨ D) → −E	
4.	F & E	⊢ −A & −C
5.	−(−A & −C)	C.A. (denial of conclusion)
6.	A ∨ C	5, DeM.
7.	B ∨ D	1,2,6, Dilemma
8.	−E	3,7, M.P.
9.	E	4, Separation
10.	E & −E	9,8, Conj.
11.	−(−A & −C) → (E & −E)	5 → 10, C.P.
12.	− −(−A & −C)	11, Reductio
13.	−A & −C	12, D.N.

(3.10) 5.

1.	(M → N) & (O → P)	
2.	−N ∨ −P	
3.	−(M & O) → B	⊢ B
4.	−B	C.A. (denial of conclusion)
5.	(−N → −M) & (−P → −O)	1, Contra. (twice)
6.	−M ∨ −O	5,2, Dilemma
7.	−(M & O)	6, DeM.
8.	B	3,7, M.P.
9.	B & −B	8,4, Conj.
10.	−B → (B & −B)	4 → 9, C.P.
11.	− −B	10, Reductio
12.	B	11, D.N.

(3.10)10.

1.	$-M$		
2.	B		
3.	$R \to Q$		
4.	$B \to R$		
5.	$(Q \,\&\, A) \to M$		$\vdash -A$
6.		A	C.A. (denial of conclusion)
7.		$-(Q \,\&\, A)$	5, 1, M.T.
8.		$-Q \vee -A$	7, DeM.
9.		$-Q$	8,6, Dis. Syll.
10.		R	4,2, M.P.
11.		Q	3,10, M.P.
12.		$Q \,\&\, -Q$	11, 9, Conj.
13.	$A \to (Q \,\&\, -Q)$		6 → 12, C.P.
14.	$-A$		13, Reductio.

(3.10)13.

1.	$A \to B$		
2.	$C \to D$		$\vdash -(A \vee C) \vee (B \vee D)$
3.		$-[-(A \vee C) \vee (B \vee D)]$	C.A. (denial of concl.)
4.		$(A \vee C) \,\&\, -(B \vee D)$	3, DeM.
5.		$A \vee C$	4, Sep.
6.		$B \vee D$	1,2,5, Dilemma
7.		$-(B \vee D)$	4, Sep.
8.		$(B \vee D) \,\&\, -(B \vee D)$	6,7, Conj.
9.	$-[-(A \vee C) \vee (B \vee D)] \to [(B \vee D) \,\&\, -(B \vee D)]$		3 → 8, C.P.
10.	$--[-(A \vee C) \vee (B \vee D)]$		9, Reductio.
11.	$-(A \vee C) \vee (B \vee D)$		10, D.N.

Exercise 3.11

(3.11) 1.

1.	$A \,\&\, B$	
2.	A	1, Separation
3.	$A \vee B$	2, Weak.
4.	$(A \,\&\, B) \to (A \vee B)$	1 → 3, C.P.

(3.11) 5.

1.	$(A \to B) \,\&\, (C \to D)$		
2.		$A \,\&\, C$	C.A.
3.		A	2, Separation
4.		C	2, Separation
5.		$A \to B$	1, Separation
6.		$C \to D$	1, Separation
7.		B	5,3, M.P.
8.		D	6,4 M.P.
9.		$B \,\&\, D$	7,8, Conj.
10.	$(A \,\&\, C) \to (B \,\&\, D)$		2 → 9, C.P
11.	$[(A \to B) \,\&\, (C \to D)] \to [(A \,\&\, C) \to (B \,\&\, D)]$		1 → 10, C.P.

(3.11) 9.	1.	│ B	
	2.	│ │ $-A$	C.A.
	3.	│ │ $-A$	Repetition
	4.	│ $-A \rightarrow -A$	2 → 3, C.P.
	5.	$B \rightarrow (-A \rightarrow A)$	1 → 4, C.P.

Exercise 3.12

(3.12) 1. It is probably easiest to construct the final "stoke" version one step at a time. The series of steps below will do the job, but it will require a bit of study.

$$\begin{aligned} A \lor B &\equiv -(-A \& -B) \\ &\equiv (-A \& -B) \mid (-A \& -B) \\ &\equiv [(-A \mid -B) \mid (-A \mid -B)] \mid (-A \;\&\; -B) \\ &\equiv [(-A \mid -B) \mid (-A \mid -B)] \mid [(-A \mid -B) \mid (-A \mid -B)] \\ &\equiv \{[(A \mid A) \mid (B \mid B)] \mid [(A \mid A) \mid (B \mid B)]\} \mid \{[(A \mid A) \mid (B \mid B)] \mid [(A \mid A) \mid (B \mid B)]\} \end{aligned}$$

(3.12) 4. $A \rightarrow B$ is, by Material Implication, logically equivalent to $-A \lor B$. Then, following a procedure similar to that used in problem 1 above, but using $-A$ instead of A, we get:

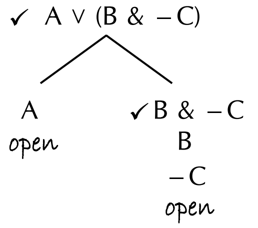
$$\{[A \mid (B \mid B)] \mid [A \mid (B \mid B)]\} \mid \{[A \mid (B \mid B)] \mid [A \mid (B \mid B)]\}$$

(3.12) 8. $(A \downarrow A) \downarrow B$

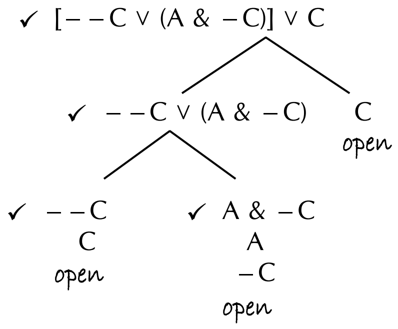
Answers to Exercises in Chapter 4

Exercise 4.1

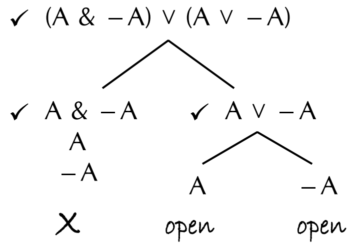
(4.1) 1.



(4.1) 3.

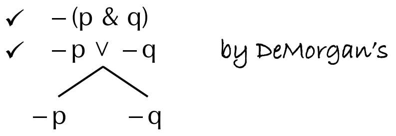


(4.1) 10.

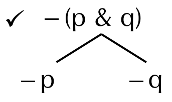


Exercise 4.2

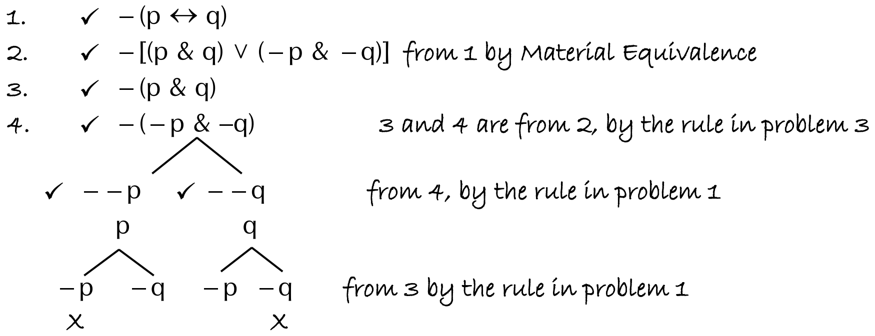
(4.2) 1.



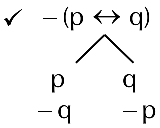
Now we may erase the mention of DeMorgan's in order to have a one-step rule:



(4.2) 5.



Now we may clean it up by eliminating the intermediate steps and the closed branches. This gives us a one-step rule:



Notice that the right branch here is not quite the same as given in problem 5, which shows

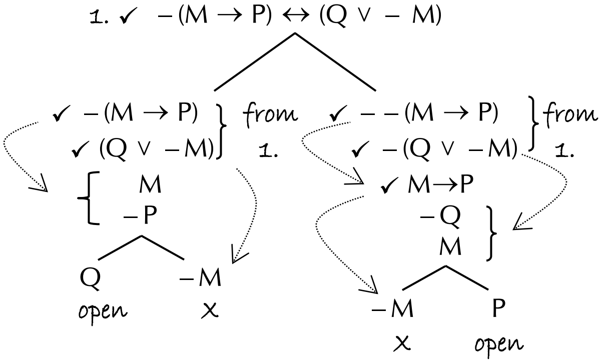
– p
q

But that is OK, since the order in which decomposition occurs is of no logical significance. It just seemed more satisfying to have some form of *p* on top of some form of *q* in both branches, but if you would rather have "positive" sentences on top of "negative" sentences, there's nothing wrong with that.

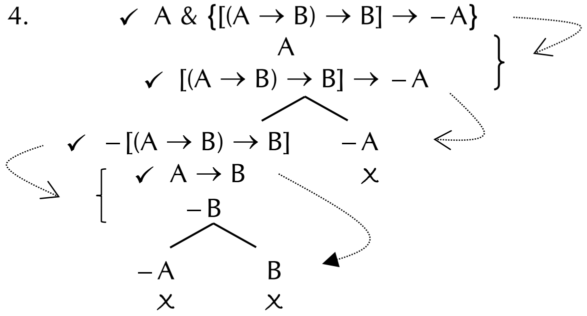
Exercise 4.3

There is often more than one way to construct a tree, so your trees may look somewhat different from mine. The important thing is whether they have the same results.

(4.3) 2.

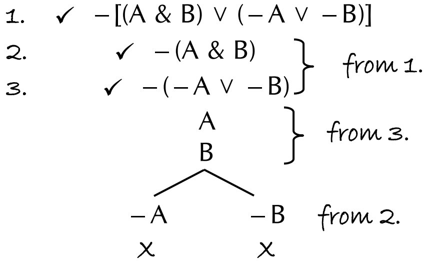


(4.3) 4.



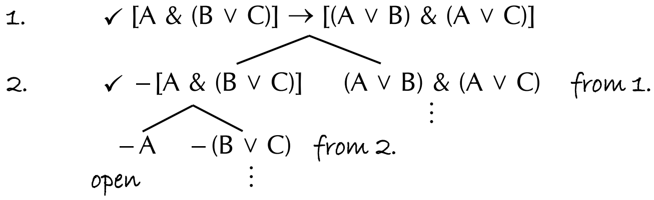
Exercise 4.4

(4.4) 1.

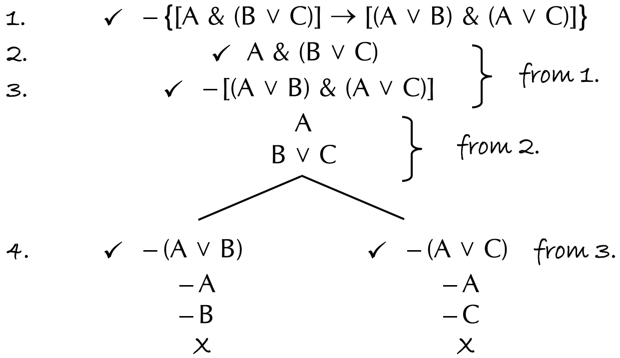


All branches close, so the sentence is contradictory.

(4.4) 3.

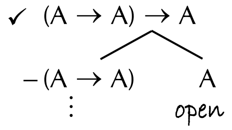


At least one branch is open, so the sentence is consistent. But that means that the sentence could be tautologous or contingent. So check the tree for the denial of the sentence:

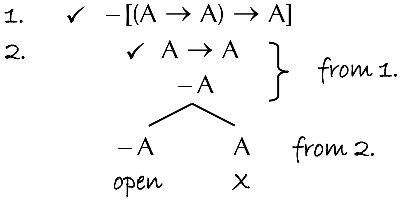


All branches close, so this sentence is inconsistent. Since this sentence is the denial of the original, the original must be the opposite of inconsistent—a tautology.

(4.4) 5.

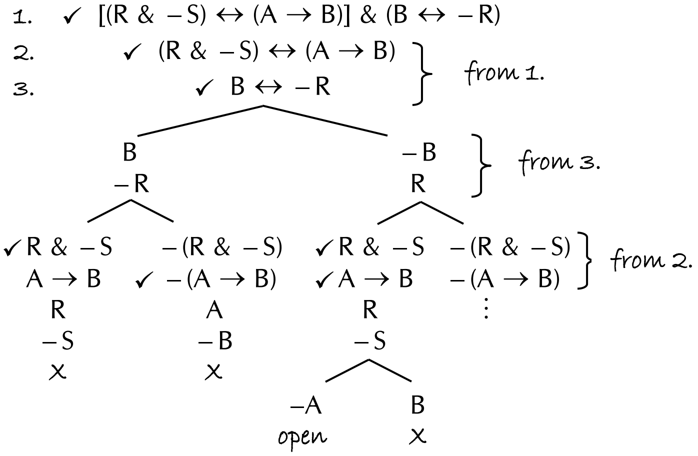


There is at least one open branch, so the sentence is consistent. But that means that the sentence could be tautologous or contingent. So check the tree for the denial:

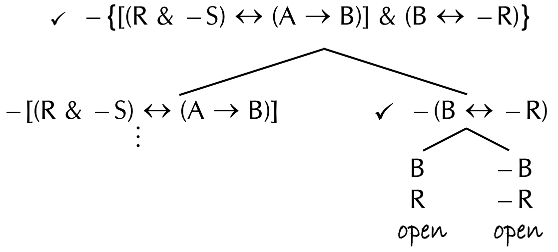


This sentence is consistent, too, so the original is contingent.

(4.4) 7.

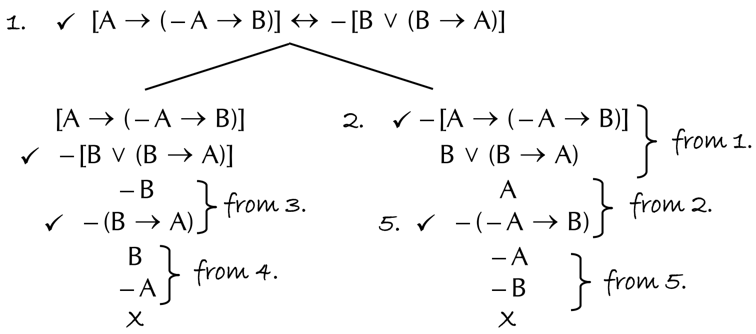


Sentence is consistent, so it's either tautologous or contingent. Check the denial:



Both original sentence and its denial are consistent; hence, the original sentence is contingent.

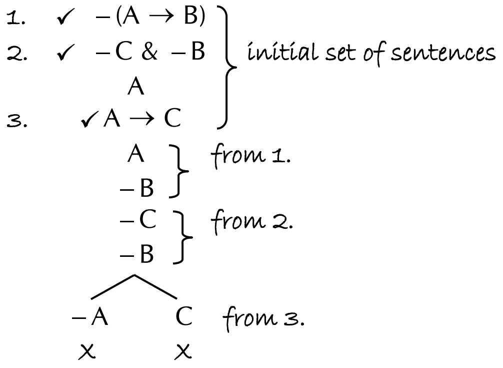
(4.4) 9.



All branches close, so the sentence is contradictory.

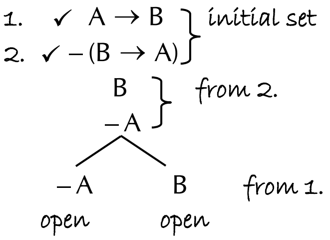
Exercise 4.5

(4.5) 1.



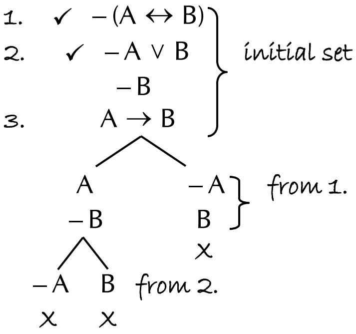
The tree closes, so the sentences are inconsistent with each other.

(4.5) 3.



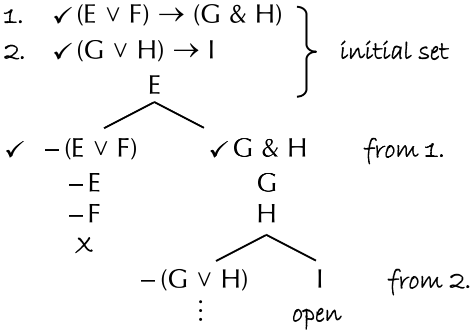
Tree does not close (i.e., there is at least one open branch), and so the sentences are consistent with each other. Any open branch provides a valuation which will make all the sentences in the original set true. From the left branch we read off: *A* is false and *B* is true. Since the right branch is also open, it, too, provides a valuation: *B* is true and *A* is false. And this just happens to be the same as the valuation given by the left branch.

(4.5) 5.



Tree closes, so the set is inconsistent.

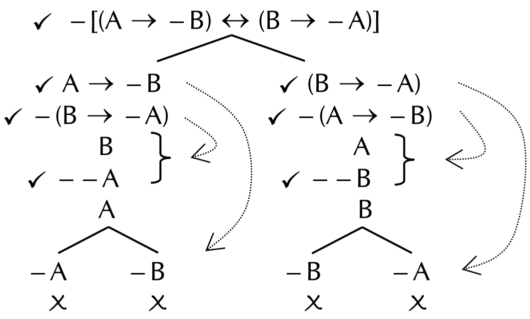
(4.5) 7.



Since there is at least one open branch, the set is consistent. A valuation of the atomic sentences which will make all the sentences in the original set true can be read off from the open branch: *I* true, *H* true, *G* true and *E* true.

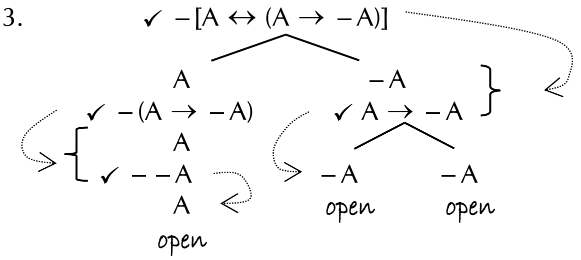
Exercise 4.6

(4.6) 1. If two sentences are equivalent, then their biconditional will be a tautology, and the tree for its denial will close:



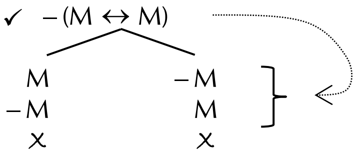
All branches do close, so the sentence at the top of the tree is inconsistent. But that sentence is the denial of the original biconditional, and so it (the original biconditional) is a tautology. Since it is a tautology, its two members have the same truth conditions, and so they are proven to be logically equivalent.

(4.6) 3.



There is at least one branch open, so the sentence at the top of the tree is consistent, which means that the original biconditional is not a tautology, which means that the two sentences of the original pair are not logically equivalent.

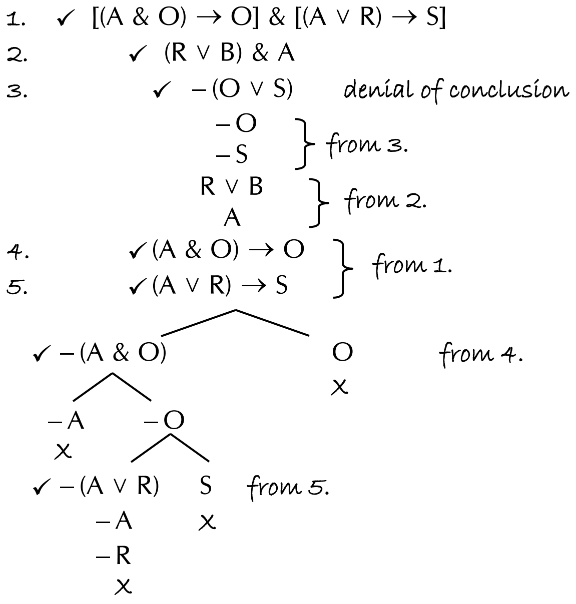
(4.6) 5.



All branches close; the two original sentences are logically equivalent.

Exercise 4.7

(4.7) 1.



All branches are closed. The argument is therefore valid.

(4.7) 5.

(J & K) → (L → M)
N → −M
−(K → −N)
1. ✓ −(J → −L)
} premises

−J denial of conclusion

J
L
} from 1.

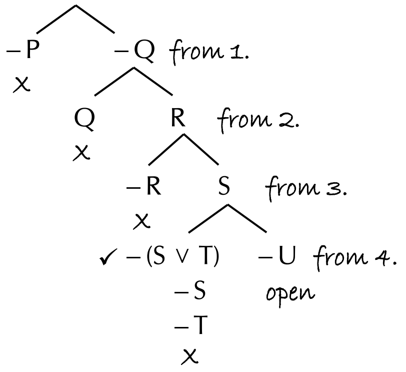
X

All branches are closed, so the argument is valid.

(4.7) 11.

1. ✓ P → −Q
2. ✓ −Q → R
3. −R ∨ S
4. ✓(S ∨ T) → −U
5. ✓ −(P → U) denial of conclusion

P
−U
} from 5.

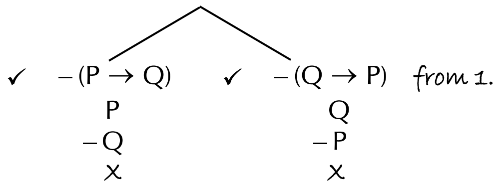


There is at least one open branch, and so the argument is invalid. A counterexample is simply the conditions which would make all the sentences in the original set true (i.e., make the premises and the denial of the conclusion true, i.e., make the premises true and the conclusion false). There is only one open branch in this tree, and it gives us a list of atomic sentences which, if true, would make the premises true and the conclusion false, namely, if *U* is false, *S* is true, *R* is true, *Q* is false, and *P* is true.

(4.7) 13.

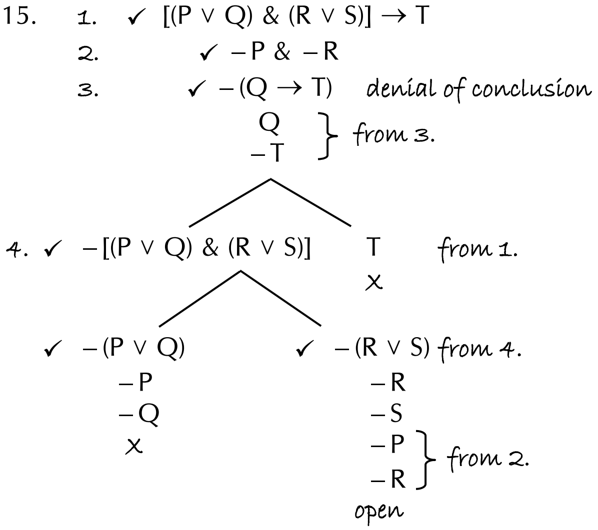
1. ✓ −[(P → Q) & (Q → P)]

P
Q denial of conclusion



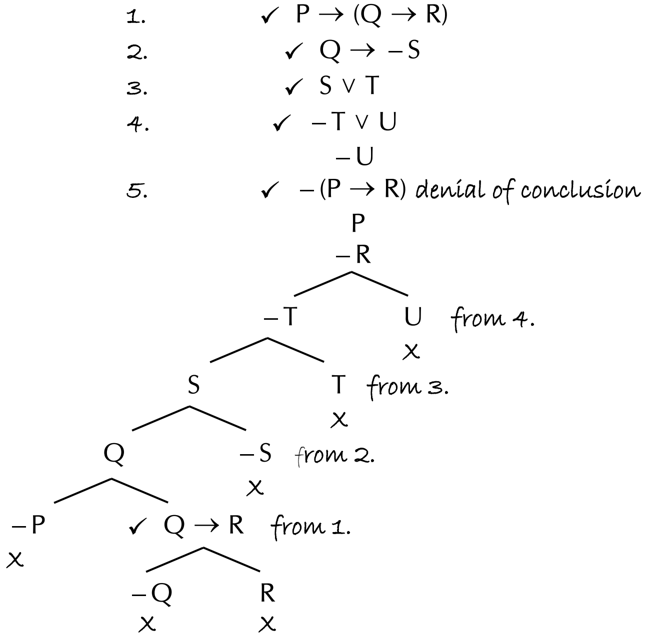
All branches are closed, so the argument is valid.

(4.7) 15.



There is at least one open branch, so the argument is invalid. A counterexample may be read off from any open branch. (In this problem, there is only one.) If *R* is false, *P* is false, *S* is false, *T* is false and *Q* is true, then the argument will have true premises and a false conclusion.

(4.7) 18.



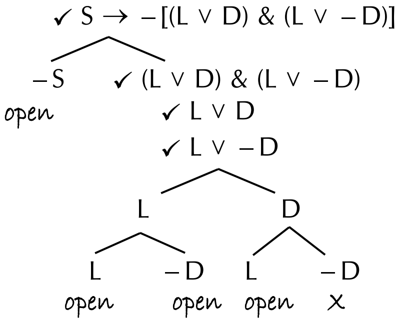
All branches are closed, so the argument is valid.

Exercise 4.8

(4.8) 1.

1.	(A → B) & (B ∨ C)	
2.	(– A ∨ B) & (B ∨ C)	1, Mat. Imp.
3.	(B ∨ – A) & (B ∨ C)	2, Comm.
4.	B ∨ (– A & C)	3, Dist.

(4.8) 5. Let's try a truth tree to start with:



From the open branches we read off: – *S* ∨ *L* ∨ (– *D* & *L*) ∨ (*L* & *D*). Now we simplify further:

1.	– S ∨ L ∨ (– D & L) ∨ (L & D)	
2.	– S ∨ L ∨ [(– D & L) ∨ (L & D)]	1, Assoc.
3.	– S ∨ L ∨ [(L & – D) ∨ (L & D)]	2, Comm.
4.	– S ∨ L ∨ [L & (– D ∨ D)]	3, Dist.
5.	– S ∨ L ∨ L	4, E.T.
6.	– S ∨ (L ∨ L)	5, Assoc.
7.	– S ∨ L	6, Idempotency

(4.8) 6. Let's try a Disjunctive Normal Form approach:

A	B	– A	B → – A	A → (B → – A)		
T	T	F	F	F		
T	F	F	T	**T**	⇨	A & – B
F	T	T	T	**T**	⇨	– A & B
F	F	T	T	**T**	⇨	– A & – B

⇩

(A & – B) ∨ (– A & B) ∨ (– A & – B)

Now to simplify:

1.	(A & – B) ∨ (– A & B) ∨ (– A & – B)	
2.	(A & – B) ∨ [(– A & B) ∨ (– A & – B)]	1, Assoc.
3.	(A & – B) ∨ [– A & (B ∨ – B)]	2, Dist.
4.	(A & – B) ∨ – A	3, E.T.
5.	– A ∨ (A & – B)	4, Comm.
6.	(– A ∨ A) & (– A ∨ – B)	5, Dist.
7.	– A ∨ – B	6, E.T.

And finally, here's a check:

A	B	−A	−B	−A ∨ −B
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

If you use a Conjunctive Normal Form approach, you'll get −*A*∨−*B* immediately, so there will be no need to simplify.

(4.8) 8.

M	Z	M → Z	M ∨ (M → Z)	−Z	[M ∨ (M → Z)] & −Z		
T	T	T	T	F	F		
T	F	F	T	T	**T**	⇨	M & −Z
F	T	T	T	F	F		
F	F	T	T	T	**T**	⇨	−M & −Z

⇩

(M & −Z) ∨ (−M & −Z)

Now to simplify that Disjunctive Normal Form expression:

1.	(M & −Z) ∨ (−M & −Z)	
2.	(−Z & M) ∨ (−Z & M)	1, Comm. (twice)
3.	−Z & (M ∨ −M)	2, Dist.
4.	−Z	3, E.T.

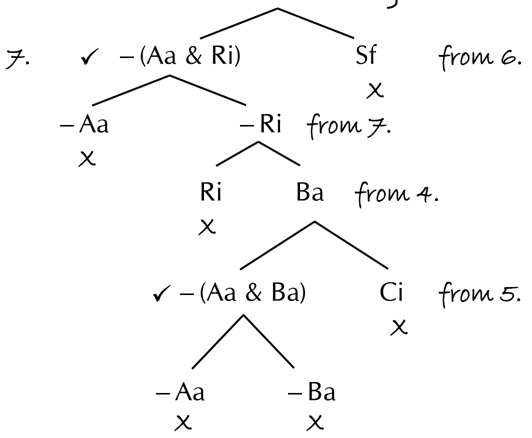
Answers to Exercises in Chapter 5

Exercise 5.1

(5.1) 4. If Socrates is Roman, then either Aristotle is Roman or Fido isn't a dog.
(5.1) 5. Aristotle is Greek if and only if Socrates is not Roman.
(5.1) 10. [(*Sj* & *Ff*) → (*Ll* ∨ *Mm*)] & [–*Sj* → (*Ml* & *Fm*)]

Exercise 5.2

(5.2) 1.
1. ✓ [(Aa & Ba) → Ci] & [(Aa & Ri) → Sf]
2. ✓ (Ri ∨ Ba) & Aa
3. ✓ –(Ci ∨ Si) *denial of conclusion*
–Ci
–Sf } *from 3.*
4. ✓ Ri ∨ Ba
Aa } *from 2.*
5. ✓(Aa & Ba) → Ci
6. ✓(Aa & Ri) → Sf } *from 1.*

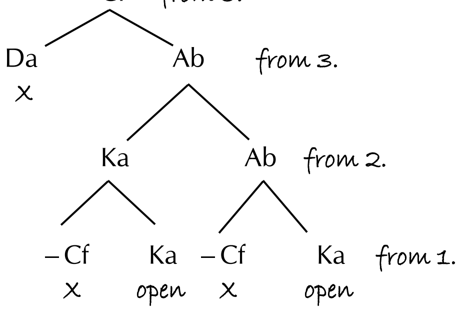


All branches close, so the argument is valid.

(5.2) 5.

1. $\checkmark\ Cf \rightarrow Ka$
2. $\checkmark\ -Ka \rightarrow Ab$
3. $\checkmark\ Da \vee Ab$
4. $\checkmark\ -(-Cf \vee Da)$ denial of conclusion
5. $\left.\begin{array}{l}\checkmark\ --Cf \\ -Da\end{array}\right\}$ from 4.

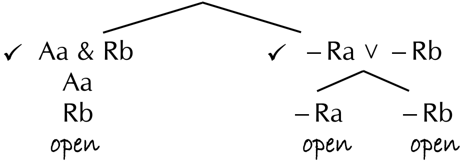
Cf from 5.



Since the tree has at least one open branch, the argument is not valid. A counterexample can be read off from the names and predicates in any open branch. The right branch, for example, gives us this (in rather formal style): Let there be a universe with three individuals, *a*, *b* and *f*. Let *a* have the property *K* but fail to have the property *D*; let *b* have the property *A*; and let *f* have the property *C*. Then the argument will have true premises and a false conclusion. A more succinct version would be: If *Ka* is true, *Ab* is true, *Cf* is true and *Da* is false (or else *–Da* is true), then the argument will have true premises and a false conclusion.

(5.2) 6.

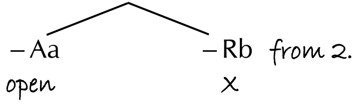
$\checkmark\ (Aa \,\&\, Rb) \vee (-Ra \vee -Rb)$



There is at least one open branch, so the sentence is consistent. But is it tautologous or contingent? Make the tree for the sentence's denial.

1. $\checkmark\ -[(Aa \,\&\, Rb) \vee (-Ra \vee -Rb)]$
2. $\checkmark\ -(Aa \,\&\, Rb)$
3. $\checkmark\ -(-Ra \vee -Rb)$ } from 1.

$\left.\begin{array}{l}Ra \\ Rb\end{array}\right\}$ from 3.



Both the original sentence and its denial are consistent, and so the original is contingent.

(5.2) 9. For some reason this sentence looks like a tautology. If it is, then its denial ought to close.

1. ✓ – {[Mp & (Lb ∨ Lc)] → [(Mp ∨ Lb) & (Mp ∨ Lc)]}
2. ✓ [Mp & (Lb ∨ Lc)] } from 1.
3. ✓ – [(Mp ∨ Lb) & (Mp ∨ Lc)] } from 1.

Mp } from 2.
Lb ∨ Lc } from 2.

✓ – (Mp ∨ Lb)
– Mp
– Lb
X

✓ – (Mp ∨ Lc) from 3.
– Mp
– Lc
X

Exercise 5.3

(5.3) 1. Gibberish. Since the variable *x* occurs in the *Gx*, it needs a quantifier, but it does not fall within the scope of the (∀*x*). Perhaps whoever wrote the sentence meant to include parentheses, thus: (∀*x*)(*Hx* → *Gx*).

(5.3) 3. OK. The quantifier has a variable in *Sx*, and the variable *x* has a matching quantifier. Notice that the name *a* also occurs within the scope of (∀*x*); but there's nothing wrong with constants appearing in quantified sentences.

(5.3) 9. OK. This is a conditional sentence. The antecedent is a quantified sentence; the consequent is another quantified sentence; and both are properly formed.

Exercise 5.4

(5.4) 1. – (∃x)(Hx → Ax)
(5.4) 3. (∃x) – Mx
(5.4) 5. Gibberish. The quantifier has no variable to quantify.
(5.4) 9. (∃x)(Ax → Bx)

Exercise 5.5

(5.5) 3. (∀x)(Ex → Px)
(5.5) 5. (∀x)(Ex → Px)
(5.5) 12. (∀x)(Ex → – Px) *or* – (∃x)(Ex & Px)
(5.5) 19. (∀x)(Ex → Px)

Note that the "something" in this sentence does not mean "at least one thing". Rather, the sentence is stating a general rule, in the same way a civil law might state a general rule applicable to everyone: "If someone breaks the speed limit, they will have to pay a fine."

Exercise 5.6

Note that there are many logically equivalent ways to express the same sentence.

(5.6) 3. $[(\forall x)(\mathrm{L}x \rightarrow \mathrm{R}x) \,\&\, (\forall x)(\mathrm{D}x \rightarrow -\mathrm{T}x)] \rightarrow (\mathrm{R}f \,\&\, \mathrm{T}f)$

(5.6) 4. $\mathrm{Pg} \,\&\, \{(\forall x)[(\mathrm{T}x \,\&\, \mathrm{N}x) \rightarrow \mathrm{H}x] \rightarrow -(\mathrm{Tg} \vee \mathrm{Ng})\}$
Note that the “only” in this sentence means “only if”: Tx and Nx only if Hx.

(5.6) 5. $\{(\forall x)[\mathrm{L}x \rightarrow (\mathrm{W}x \rightarrow \mathrm{C}x)] \,\&\, (\mathrm{La} \,\&\, \mathrm{Wa})\} \rightarrow \mathrm{Ca}$

(5.6) 9. $[(\forall x)(\mathrm{P}x \rightarrow \mathrm{M}x) \,\&\, \mathrm{Pr}] \rightarrow \mathrm{Mr}$

(5.6) 16. All ostriches are not larger than an elephant. (Or: No ostrich is larger than an elephant.)
Note the different placement of the denial sign as compared to problem 17.

(5.6) 17. Not all ostriches are larger than an elephant.
Note the different placement of the denial sign as compared to problem 16.

(5.6) 19. Olga is a horse if and only if all ostriches are horses.

Answers to Exercises in Chapter 6

Exercise 6.1

(6.1) 1. *Gx*: *x* is a good philosopher. *Hx*: *x* is honored by its readers. *Lx*: *x* is a logician.

1. $(\forall x)(Gx \to Hx)$ } premises
2. $(\forall x)(Lx \to Gx)$
3. ✓ $-(\forall x)(Lx \to Hx)$ denial of conclusion
4. ✓ $(\exists x)-(Lx \to Hx)$ from 3 by duality rules

✓ $-(La \to Ha)$ from 4 by E.I. using a new name, a

La

$-Ha$

✓ $La \to Ga$ from 2 by U.I. using the a

$-La$ $\qquad$ Ga

X $\qquad$ ✓ $Ga \to Ha$ from 1 by U.I. using the a

$-Ga$ $\qquad$ Ha

X $\qquad$ X

Tree closes; argument is valid.

(6.1) 3. *Ex*: *x* is an Englishman. *Bx*: *x* is Brazilian. *Hx*: *x* is Hindu

1. $(\forall x)(Ex \to -Bx)$ } premises
2. $(\forall x)(Bx \to -Hx)$
3. ✓ $-(\forall x)(Ex \to -Hx)$ denial of conclusion
4. ✓ $(\exists x)-(Ex \to -Hx)$ from 3 by duality rule

✓ $-(Ea \to -Ha)$ from 4 by E.I. using a new name, a

Ea

✓ $--Ha$

Ha

✓ $Ea \to -Ba$ from 1 by U.I. using the a

$-Ea$ $\qquad$ $-Ba$

X $\qquad$ ✓ $Ba \to -Ha$ from 2 by U.I. using the a

$-Ba$ $\qquad$ $-Ha$

open $\qquad$ X

Invalid. Counterexample: Let there be a universe of one thing, *a*, such that *a* is *H* and *E* but not *B*. Then the argument will have true premises and a false conclusion.

(6.1) 5. *Tx*: *x* is a thief. *Lx*: *x* should expect to be let off easy. *Mx*: *x* is a person who makes restitution to the victims after having taken possessions....

1. $(\forall x)(Tx \rightarrow -Lx)$
2. $(\forall x)(-Mx \rightarrow Tx)$
3. ✓ $-(\forall x)(-Mx \rightarrow -Lx)$ denial of conclusion
4. ✓ $(\exists x)-(-Mx \rightarrow -Lx)$ from 3 by duality rule

✓ $-(-Ma \rightarrow -La)$ from 4 by E.I. using a new name, a
$-Ma$
✓ $--La$
La
✓ $Ta \rightarrow -La$ from 1 by U.I. using the a
$-Ta$ $-La$
from 2 by U.I. using the a $-Ma \rightarrow Ta$ X
✓ $--Ma$ Ta
Ma X
X

Tree closes; argument is valid.

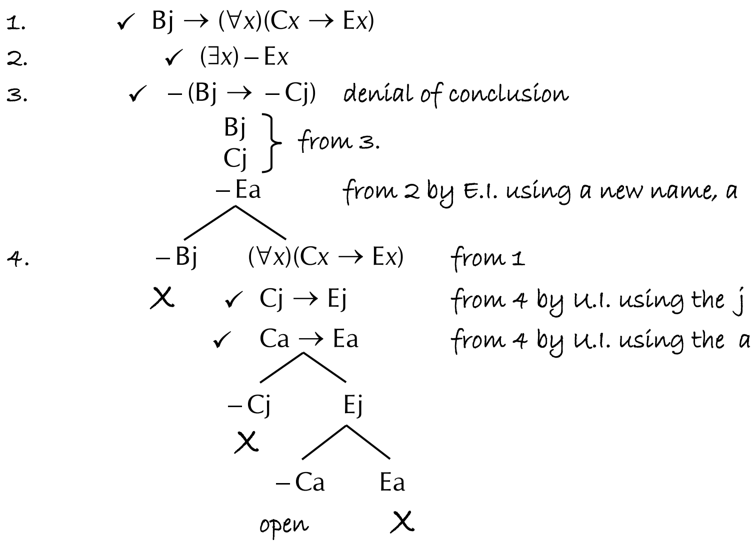
(6.1) 9. *F*x: *x* is a farmer. *H*x: *x* is happy. *R*x: *x* is a rich man.

1. $(\forall x)(Hx \rightarrow Fx)$
2. ✓ $(\exists x)(Rx \ \& \ -Fx)$
3. ✓ $-(\exists x)(Rx \ \& \ -Hx)$ denial of conclusion
4. $(\forall x)-(Rx \ \& \ -Hx)$ from 3 by duality rule

✓ $Ra \ \& \ -Fa$ from 2 by E.I. using a new name, a
Ra
$-Fa$
✓ $Ha \rightarrow Fa$ from 1 by U.I. using the a
$-Ha$ Fa
from 4 by U.I. using the a ✓ $-(Ra \ \& \ -Ha)$ X
$-Ra$ ✓ $--Ha$
X Ha
X

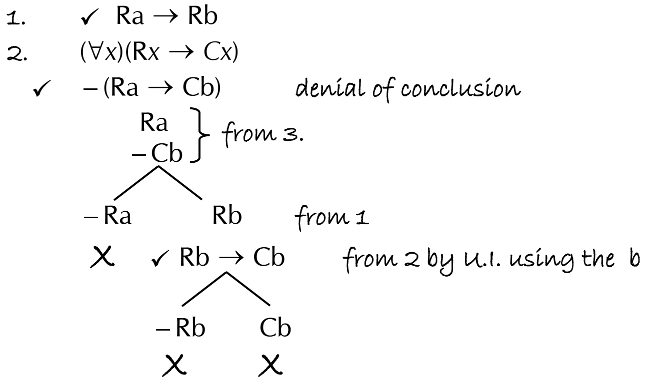
Valid.

(6.1) 13.



Invalid. Counterexample: Let there be a universe of two things, *a* and *j*, such that *a* fails to have the properties *C* and *E*, and *j* has the properties *E*, *C* and *B*. Then the argument will have true premises and a false conclusion.

(6.1) 15.



Valid.

Exercise 6.2

(6.2) 1. *Mx*: *x* is a mathematician. *Tx*: *x* is trained in calculus.

$(\forall x)(Mx \rightarrow Tx)$

(6.2) 3. *Tx*: *x* takes the initiative. *Rx*: *x* is rewarded.

$(\forall x)(Tx \rightarrow Rx)$

Exercise 6.3

Note: There are often many ways to translate a sentence. Your translations may differ from those given below.

(6.3) 3. *Bx*: *x* is a bat. *Wx*: *x* is winged. *Ax*: *x* is an animal.

$(\forall x)[Bx \rightarrow (Wx \,\&\, Ax)]$

(6.3) 6. *Fx*: *x* is a fish. *Sx*: *x* is a shark. *Kx*: *x* is kind to children.

$(\forall x)[(Fx \,\&\, -Sx) \rightarrow Kx]$

(6.3) 10. *Hx*: *x* is happy. *Fx* : *x* is free.
(For convenience, restrict the universe of discourse to men.)

$(\forall x)(Hx \rightarrow Fx)$

In each of the following answers, alternative translations are given. Still other translations would be quite all right, provided they have the same meaning.

(6.3) 11. Non-equiangular triangles are not equilateral.
If a triangle is not equiangular, then it is not equilateral.
Triangles must be equilateral in order to be equiangular.

(6.3) 13. Rectangles which are squares have equal sides.
In order to be a square, a rectangle has to have equal sides.

(6.3) 15. Popular girls are cheerful.
All popular girls are cheerful.

(6.3) 19. If no one was murdered then no one is guilty.
Nobody was murdered only if nobody is guilty.

Exercise 6.4

(6.4) 1. *Px*: *x* is a person. *Rx*: *x* has rights. *Cx*: *x* is a computer. *h*: Hal.

1. $(\forall x)(Rx \rightarrow Px)$ } premises
2. ✓ $Ch \,\&\, -Ph$

✓ $--Rh$ denial of conclusion
Rh
Ch } from 2.
$-Ph$
3. ✓ $Rh \rightarrow Ph$ from 1. by U.I. using the h

$-Rh$ Ph from 3.
X X

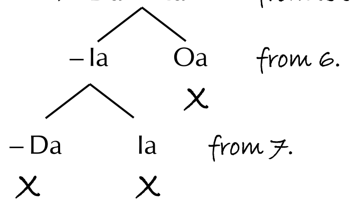
Valid

(6.4) 3. *Ix*: *x* inhibits your ability to deal effectively with.... *Ox*: *x* ought to be avoided. *Dx*: *x* is drinking yourself into an insensitive stupor.

1. $(\forall x)(Ix \to Ox)$ } premises
2. $(\forall x)(Dx \to Ix)$ }
3. ✓ $-(\forall x)(Dx \to Ox)$ denial of conclusion
4. ✓ $(\exists x)-(Dx \to Ox)$ from 3. by duality rules
5. ✓ $-(Da \to Oa)$ from 4. by E.I. using a new name, a.

Da }
$-Oa$ } from 5.

6. ✓ $Ia \to Oa$ from 1. by U.I. using the a.
7. ✓ $Da \to Ia$ from 2 by U.I. using the a.



Valid

(6.4) 5. *Dx*: *x* is a dentist. *Mx*: *x* is a monkey. *Ax*: *x* is an animal. *Ex*: *x* can extract teeth.

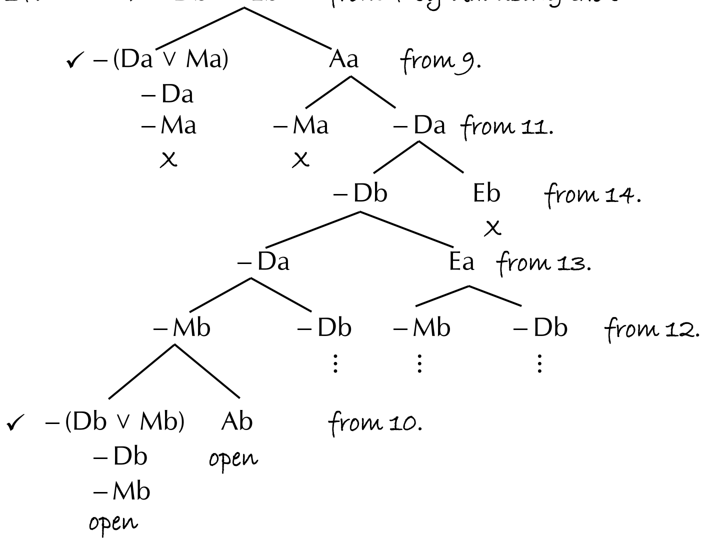
1. $(\forall x)[(Dx \vee Mx) \rightarrow Ax]$
2. ✓ $(\exists x)(Ax \;\&\; -Ex)$ or: $-(\forall x)(Ax \rightarrow Ex)$
3. $(\forall x)(Mx \rightarrow -Dx)$
4. $(\forall x)(Dx \rightarrow Ex)$
5. ✓ $-(\forall x)(Mx \rightarrow -Ex)$ denial of conclusion
6. ✓ $(\exists x)-(Mx \rightarrow -Ex)$ from 5 by duality rule
7. ✓ $-(Ma \rightarrow -Ea)$ from 6 by E.I. using a new name, a

Ma
✓ $--Ea$ } from 7.
Ea

8. ✓ $Ab \;\&\; -Eb$ from 2 by E.I. using a new name, b

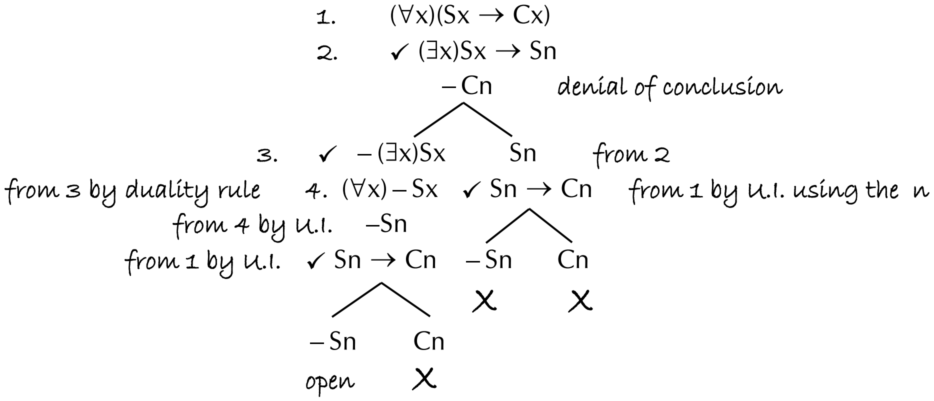
Ab
$-Eb$ } from 8.

9. ✓ $(Da \vee Ma) \rightarrow Aa$ from 1 by U.I. using the a
10. ✓ $(Db \vee Mb) \rightarrow Ab$ from 1 by U.I. using the b
11. ✓ $Ma \rightarrow -Da$ from 3 by U.I. using the a
12. ✓ $Mb \rightarrow -Db$ from 3 by U.I. using the b
13. ✓ $Da \rightarrow Ea$ from 4 by U.I. using the a
14. ✓ $Db \rightarrow Eb$ from 4 by U.I. using the b



Invalid. Counterexample (from the left-most branch): Let there be a universe of two entities, *a* and *b*, such that *a* is *A* and *E* and *M* but not *D*, and *b* is *A* but not M nor *D* nor *E*. Then the argument will have true premises and a false conclusion.

(6.4) 9. *Sx*: *x* is a good science fiction writer. *Cx*: *x* is creative. *n*: Larry Niven.



Invalid. Counterexample: Let there be a universe of one thing, *n*, such that *n* is neither *S* nor *C*. Then the argument will have true premises and a false conclusion.

Answers to Exercises in Chapter 7

Exercise 7.1

(7.1) 3. The symbolic version indicates that Elizabeth is a parent of someone, and that someone is a parent of Alice. We can express that in more ordinary English as: Elizabeth is a grandparent of Alice. (Note that we cannot say, based on the information provided in this sentence, that Elizabeth is Alice's grand*mother*, nor that Alice is Elizabeth's grand*daughter*, because the gender of Elizabeth and Alice is not given.)

(7.1) 7. Roughly: Fred is male and not a sibling of Elizabeth. Better is: Fred is not Elizabeth's brother.

(7.1) 8. Roughly: It is false that there is a parent of Alice. Better is: Alice has no parents. Or: Alice is an orphan.

(7.1) 11. Brbs

(7.1) 12. Egm

Exercise 7.2

(7.2) 1. Somebody is the father of Fred; that somebody has a sibling; and that sibling is a parent of Alice. *Or more succinctly:* Fred and Alice are cousins.

(7.1) 3. No one is his/her own parent.

(7.1) 5. There is someone such that anyone who is female is not a sibling of that someone. *Or more succinctly:* There is someone who has no sister.

(7.1) 7. Someone is a male sibling of Alice. *Or better yet:* Alice has a brother.

Exercise 7.3

(7.3) 1.

1. $(\forall x)(\exists y)(Gx \to Myx)$

   Ga

2. ✓ $-(\exists x)Mxa$ denial of conclusion
3. $(\forall x)-Mxa$ from 2. by duality rules
4. ✓ $(\exists y)(Ga \to Mya)$ from 1 by U.I. using the a
5. ✓ $Ga \to Mba$ from 4 by E.I. using a new name, b

$-Ga$ Mba from 5

X $-Mba$ from 3 by U.I. using the b

X

All branches close. Argument is valid.

(7.3) 3.

1. (∀x)[Rx → (∃x)Lyx]

2. ✓ – (Rc → Lec) *denial of conclusion*

Rc } *from 2*
– Lex }

3. ✓ Rc → (∃y)Lyc *from 1 by U.I. using the c.*

✓ – Rc 4. ✓ (∃y)Lyc *from 3.*

X Ldc *from 4 by E.I. using a new name, d*

Rd → (∃y)Lyd *from 1 by U.I. using the d*

⋮

It's pretty clear that we have an infinite branch here. Since the tree will not close, the argument is invalid. A counterexample is available from any open branch. If *d* stands in the relation *L* to *c*, and if *e* does not stand in the relation *L* to *c*, and if *c* has the property *R*, then the argument will have true premises and a false conclusion.

(7.3) 7.

1. Scl

2. ✓ (∃x)Sxl → Bl

– Bl *denial of conclusion*

3. ✓ – (∃x)Sxl Bl *from 2.*

4. *from 3 by duality* (∀x) – Sxl X

– Scl *from 4 by U.I. using the c*

X

All branches are closed, so the argument is valid.

(7.3) 13.

1. (∀x)[Txi → (∃y)(Txy & Tyg)]

✓ – [Tki → (∃x)Txg] *denial of conclusion*

Tki }
✓ – (∃x)Txg }

2. (∀x) – Txg

✓ Tki → (∃y)(Tky & Tyg) *from 1 by U.I. using the k*

– Tki ✓ (∃y)(Tky & Tyg)

X ✓ Tka & Tag

{ Tka
{ Tag

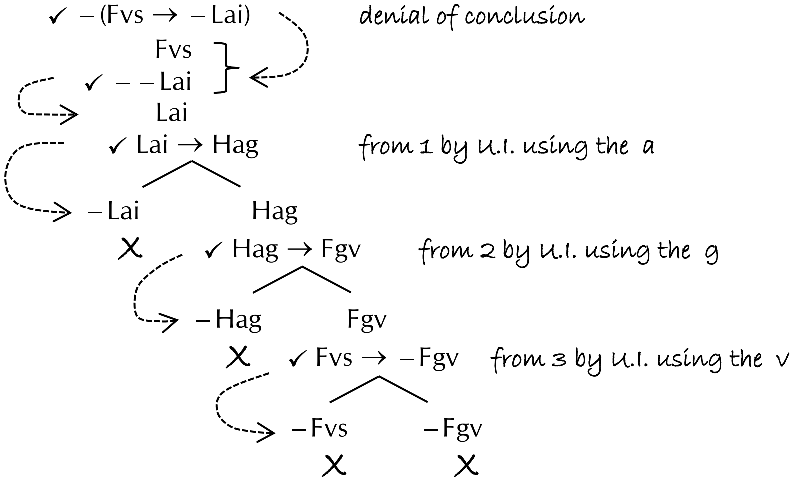
– Tag *from 2 by U.I. using the a*

X

All branches are closed, so the argument is valid.

(7.3) 15.

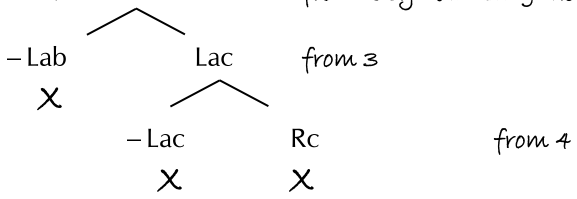
1. (∀x)(Lxi → Hxg)
2. (∀x)(Hax → Fxv)
3. (∀x)(Fxs → −Fgx) or −(∃x)(Fxs & Fgx)



All branches close, so the argument is valid.

(7.3) 18.

Lab
1. (∀x)(Lxb → Lxc)
2. (∀x)(Lax → Rx)
−Rc denial of conclusion
3. ✓ Lab → Lac from 1 by U.I. using the a
4. ✓ Lac → Rc from 2 by U.I. using the c



All branches are closed, so the argument is valid.

Exercise 7.4

(7.4) 1. Alice has a parent who is not Fred.
(7.4) 3. Males and females are different (i.e., non-identical).
(7.4) 7. Fred is the only male in the universe. *Or:* Fred is male, and nobody else is.

(7.4) 11.

1. $(\forall x)(Rx \to Gx)$ premises
Ra
2. $a = b$
$-Gb$ denial of conclusion
3. ✓ $Ra \to Ga$ from 1 by U.I. using the a
$-Ra$ 4. Ga from 3
X Gb from 4 and 2
X

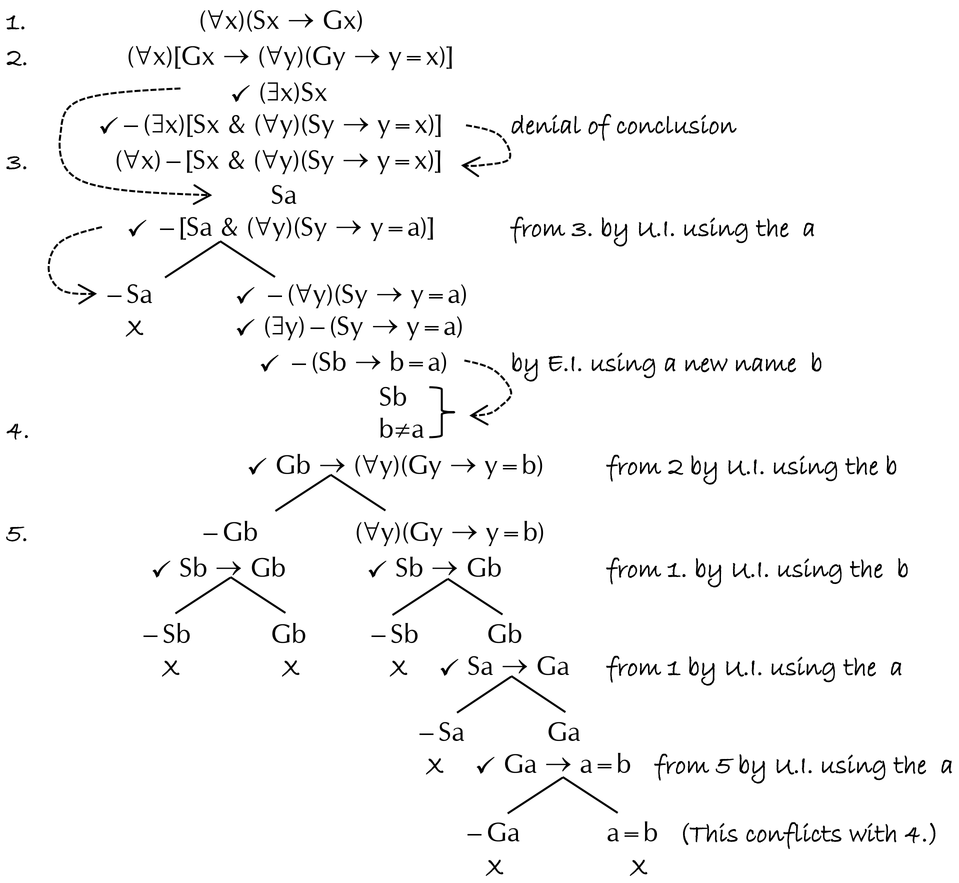
All branches closed. Argument is valid.

(7.4) 13.

1. ✓ $(\exists x)[Rx \,\&\, (\forall y)(Ry \to x = y)$ premises
Ra
2. ✓ $-(Rb \to a = b)$ denial of conclusion
Rb from 2
3. $-(a = b)$
4. ✓ $Rc \,\&\, (\forall y)(Ry \to c = y)$ from 1 by E.I. using a new name, c
Rc from 4
5. $(\forall y)(Ry \to c = y)$
6. ✓ $Ra \to c = a$ from 5 by U.I. using the a
7. $-Ra$ $c = a$ from 6
8. X ✓ $Rb \to c = b$ from 5 by U.I. using the b
9. $-Rb$ $c = b$ from 8
X $a = b$ from 7 and 9
X (contradicts line 3)

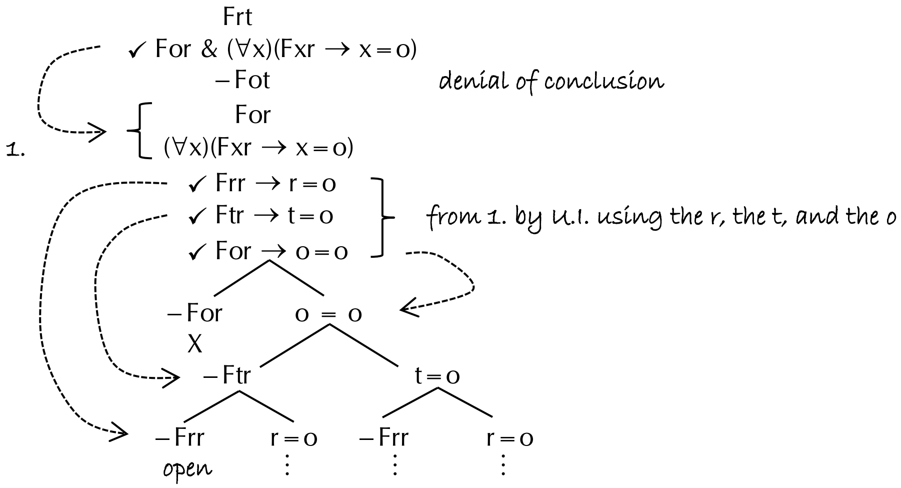
All branches closed, so argument is valid.

(7.4) 16.



All branches close, so the argument is valid.

(7.4) 17.



Not all branches close, so the argument is invalid.
Counterexample: Let Frr be false, Ftr be false, o=o [a necessary truth], For be true, Fot be false, and Frt be true; then the argument will have true premises and a false conclusion.

Exercise 7.5

(7.5) 1. If identity is transitive, then this must be a tautology: $(\forall x)(\forall y)(\forall z)[(x=y \ \& \ y=z)\rightarrow x=z]$. If that sentence is a tautology, then the tree for its denial will close. And it does:

1. ✓ $(\forall x)(\forall y)(\forall z)[(x=y \ \& \ y=z) \rightarrow x=z]$
2. ✓ $(\exists x)(\exists y)(\exists z)-[(x=y \ \& \ y=z) \rightarrow x=z]$ from 1 by duality rules (thrice)
3. ✓ $(\exists y)(\exists z)-[(a=y \ \& \ y=z) \rightarrow a=z]$ from 2 by E.I. using a new name, a
4. ✓ $(\exists z)-[(a=b \ \& \ b=z) \rightarrow a=z]$ from 3 by E.I. using a new name, b.
5. ✓ $-[(a=b \ \& \ b=c) \rightarrow a=c]$ from 4 by E.I. using a new name, c
6. ✓ $(a=b \ \& \ b=c)$ } from 5
7. $a\neq c$
8. $a=b$ } from 6
9. $b=c$
$b\neq c$ from 7 and 8.
X (Contradicts 9.)

If identity is reflexive, then this must be a tautology: $(\forall x)(x=x)$. And it is, because the tree for its denial closes:

✓ $-(\forall x)(x=x)$
✓ $(\exists x)-(x=x)$ duality rule
$-(a=a)$ by E.I. using a new name a
X (self-contradiction)

Finally, if identity is symmetrical, then this will be a tautology: $(\forall x)(\forall y)(x=y \rightarrow y=x)$. And it is, because the tree for its denial closes:

1. ✓ $-(\forall x)(\forall y)(x=y \rightarrow y=x)$
2. ✓ $(\exists x)(\exists y)-(x=y \rightarrow y=x)$ from 1 by duality rule (twice)
3. ✓ $(\exists y)-(a=y \rightarrow y=a)$ from 2 by E.I. using a new name a
4. ✓ $-(a=b \rightarrow b=a)$ from 3 by E.I. using a new name b
5. $a=b$
$-(a=b)$
X (contradicts 5.)

(7.5) 2. *Transitive*, because if *x* is the brother of *y* and *y* is the brother of *z*, then *x* must be the brother of *z* as well. *Irreflexive*, because one cannot be one's own brother. *Non-symmetrical*, because if *x* is the brother of *y*, then *y* could be, but need not be, a brother of *x*. That is, *y* would be the brother of *x* if *y* was male, but not if *y* was female.

(7.5) 5. *Reflexive*, because a thing surely resembles itself (although it sounds a bit odd to say so). *Non-transitive*, because if *x* resembles *y* and *y* resembles *z*, then *x* might very well resemble *z*, but that need not always be the case: perhaps it could happen that *x* faintly resembles *y*, and *y* faintly resembles *z*, yet *x* and *z* have nothing in common. *Symmetric*, because if something resembles some other thing (in some respect or other), then that second thing surely resembles the first in the same respect.

(7.5) 8. *Non-Reflexive*, because one can, but need not, surprise oneself. *Non-transitive*, because if I surprise you, and you surprise Fred, it doesn't follow that I surprise Fred. *Non-symmetrical*, because if I surprise you, then you might or might not surprise me back.

Answers to Exercises in Chapter 8

Exercise 8.1

(8.1) 1.

2. C.A. (denial of conclusion)
3. 2, Duality
4. 3, E.I. with a new name *b*
5. 4, Material Implication
6. 5, DeMorgan's
7. 6, Separation
8. 1, 8, Conjunction
9. 2→8, C.P.
10. 9, Reductio.
11. 10, Double Negation.

(8.1) 3.

1.	(∀y)[(Ay & By) → Cy]	
2.	−Cb	⊢ −(Ab & Bb)
3.	(Ab & Bb) → Cb	1, U.I., using the *b*
4.	−(Ab & Bb)	3, 2, Mod. Pon.

Exercise 8.2

(8.2) 1.

1.	(∀x)(Ax → Bx)	
2.	(∃x)−Bx	⊢ −Aa
3.	−Ba	2, E.I., using a new name, *a*
4.	Aa → Ba	1, U.I., using the *a*
5.	−Aa	4, 3, Mod. Toll.

(8.2) 5.

1.	(∀x)(Mx → Lx)	
2.	−(∀x)(Rx → Lx)	⊢ (∃x)(Rx & −Mx)
3.	(∃x)−(Rx → Lx)	2, Duality
4.	−(Ra → La)	3, E.I. using a new name *a*
5.	−(−Ra ∨ La)	4, Mat. Imp.
6.	Ra & −La	5, DeMorgan's
7.	Ra	6, Sep.
8.	−La	6, Sep.
9.	Ma → La	1, U.I. using the *a*
10.	−Ma	9,8, Mod. Toll.
11.	Ra & −Ma	7, 10, Conjunction
12.	(∃x)(Rx & −Mx)	11, E.G.

(8.2) 6.	1.	$(\forall x)(Rx \rightarrow -Sx)$	$\vdash -(\exists x)(Rx \,\&\, Sx)$
	2.	$(\exists x)(Rx \,\&\, Sx)$	C.A. (denial of conclusion)
	3.	$Ra \,\&\, Sa$	2, E.I. using a new name *a*
	4.	Ra	3, Sep.
	5.	Sa	3, Sep.
	6.	$Ra \rightarrow -Sa$	1, U.I. using the *a*
	7.	$-Sa$	6, 4, Mod. Pon.
	8.	$Sa \,\&\, -Sa$	5, 7, Conj.
	9.	$(\exists x)(Rx \& Sx) \rightarrow Sa \& -Sa$	2→8, C.P.
	10.	$-(\exists x)(Rx \,\&\, Sx)$	9, Reductio

(8.2) 10.	1.	$(\forall x)(Ax \rightarrow -Cx)$	
	2.	$(\forall y)(By \vee Cy)$	
	3.	$(\forall x)Ax$	$\vdash (\forall y)By$
	4.	$A1 \rightarrow -C1$	1, U.I., choosing quasi-variable *1*
	5.	$A1$	3, U.I., using the *1*
	6.	$-C1$	4, 5, Mod. Pon.
	7.	$B1 \vee C1$	2, U.I. using the *1*
	8.	$B1$	7, 6, Dis. Syll.
	9.	$(\forall y)By$	8, U.G.

(8.2) 12.	1.	$(\forall x)(Lx \rightarrow Mx)$	
	2.	$(\exists x)Lx \vee (\exists x)Mx$	$\vdash (\exists x)Mx$
	3.	$-(\exists x)Mx$	C.A. (denial of conclusion)
	4.	$(\exists x)Mx$	2, 3, Dis. Syll.
	5.	La	4, E.I., choosing a new name *a*
	6.	$La \rightarrow Ma$	1, U.I., using the *a*
	7.	Ma	6, 5, Mod. Toll.
	8.	$(\forall x)-Mx$	3, Duality
	9.	$-Ma$	8, U.I., using the *a*
	10.	$Ma \,\&\, -Ma$	7, 9, Conj.
	11.	$-(\exists x)Mx \rightarrow Ma \,\&\, -Ma$	3→10, C.P.
	12.	$(\exists x)Mx$	11, Reductio

(8.2) 14.	1.	$(\forall x)[(Bx \vee Cx) \rightarrow Sx]$	$\vdash (\forall x)[(Cx \,\&\, -Ax) \rightarrow Sx]$
	2.	$(B1 \vee C1) \rightarrow S1$	1, U.I., choosing a quasi-variable, *1*
	3.	$-(B1 \vee C1) \vee S1$	2, Mat. Imp.
	4.	$(-B1 \,\&\, -C1) \vee S1$	3, DeMorgan's
	5.	$(-B1 \vee S1) \,\&\, (-C1 \vee S1)$	4, Dist.
	6.	$-C1 \vee S1$	5, Sep.
	7.	$(-C1 \vee S1) \vee A1$	6, Weak.
	8.	$(-C1 \vee A1) \vee S1$	7, Assoc., Comm.
	9.	$-(-C1 \vee A1) \rightarrow S1$	8, Mat. Imp.
	10.	$(C1 \,\&\, -\ A1) \rightarrow S1$	9, DeMorgan's
	11.	$(\forall x)[(Cx \,\&\, -Ax) \rightarrow Sx]$	10. U.G.

(8.2) 15.	1.	−(∃x)(Mxa & −Oxb)	
	2.	−(∃x)(Dxc & Dbx)	
	3.	(∀x)(Oex → Dxf)	⊢ −(Mea & Dfc)
	4.	(∀x)−(Mxa & −Oxb)	1, Duality
	5.	(∀x)−(Dxc & Dbx)	2, Duality
	6.	−(Mea & −Oeb)	4, U.I., using the e
	7.	−Mea ∨ Oeb	6, DeMorgan's
	8.	Mea → Oeb	7, Mat. Imp.
	9.	Oeb → Dbf	3, U.I., using the e
	10.	Mea → Dbf	8, 9, Hyp. Syll.
	11.	−(Dfc & Dbf)	5, U.I. using the *f*
	12.	−Dfc ∨ −Dbf	11, DeMorgan's
	13.	Dfc → −Dbf	12, Mat. Imp.
	14.	Dbf → −Dfc	13, Contraposition
	15.	Mea → −Dfc	10, 14, Hyp. Syll.
	16.	−Mea ∨ −Dfc	15, Mat. Imp.
	17.	−(Mea & Dfc)	16, DeMorgan's

(8.2) 18.	1.	(∃x)Gx → (∃x)(Ax & Kx)	
	2.	(∃x)(Kx ∨ Lx) → (∀x)Mx	⊢ (∀x)(Gx → Mx)
	3.	−(∀x)(Gx → Mx)	C.A. (denial of conclusion)
	4.	(∃x)−(Gx →Mx)	3, Duality
	5.	−(Ga →Ma)	4, E.I. using a new name *a*
	6.	−(−Ga ∨ Ma)	5, Mat. Imp.
	7.	Ga & −Ma	6, DeMorgan's
	8.	Ga	7, Sep.
	9.	(∃x)Gx	8, E.G.
	10.	(∃x)(Ax & Kx)	1, 9, Mod. Pon.
	11.	−Ma	7, Sep.
	12.	(∃x)−Mx	11, E.G.
	13.	−(∀x)Mx	12, Duality
	14.	−(∃x)(Kx ∨ Lx)	2, 13, Mod. Toll.
	15.	(∀x)−(Kx ∨ Lx)	14, Duality
	16.	Ab & Kb	10, E.I. choosing a new name *b*
	17.	Kb	16, Sep.
	18.	−(Kb ∨ Lb)	15, U.I. using the *b*
	19.	−Kb & −Lb	18, DeMorgan's
	20.	−Kb	19, Sep.
	21.	Kb & −Kb	17, 20, Conj.
	22.	−(∀x)(Gx → Mx) → (Kb & −Kb)	3→21, C.P.
	23.	(∀x) (Gx → Mx)	22, Reductio

(8.2) 20.

1.	(∀x)[Zx → (∃y)(Zy & Rxy)]	
2.	(∃x){Zx & (∀y)[(Zy & Rxy) → Cxy]}	⊢ (∃x)(∃y)[(Zx & Zy) → Cxy]
3.	Za & (∀y)[(Zy & Ray) → Cay]	2, E.I. choosing a new name *a*
4.	Za	3, Sep.
5.	Za → (∃y)(Zy & Ray)	1, U.I. using the *a*
6.	(∃y)(Zy & Ray)	4, 5, Mod. Pon.
7.	Zb & Rab	6, E.I., choosing a new name *b*
8.	(∀y)[(Zy & Ray) → Cay]	3, Sep.
9.	(Zb & Rab) → Cab	8, U.I. using the *b*
10.	Cab	9, 7, Mod. Pon.
11.	−(∃x)(∃y)[(Zx & Zy) → Cxy]	C.A. (denial of conclusion)
12.	(∀x)(∀y)−[(Zx & Zy) → Cxy]	11, Duality
13.	(∀y)−[(Za & Zy) → Cay]	12, U.I. using the *a*
14.	−[(Za & Zb) → Cab]	13, U.I. using the *b*
15.	−[−(Za & Zb) ∨ Cab]	14, Mat. Imp.
16.	(Za & Zb) & −Cab	15, DeMorgan's
17.	−Cab	16, Sep.
18.	Cab & −Cab	10, 17, Conj.
19.	−(∃x)(∃y)[(Zx&Zy)→Cxy]→(Cab&−Cab)	11→18, C.P.
20.	(∃x)(∃y)[(Zx&Zy)→Cxy]	19, Reductio

Exercise 8.3

(8.3) 2.

1.	(∀x)(Ax → Bx)	
2.	(∀x)(Bx → Mx)	
3.	Aa & −Mb	⊢ a≠b
4.	Aa → Ba	1, U.I. using the *a*
5.	Ba → Ma	2, U.I. using the *a*
6.	Aa →Ma	4, 5, Hyp. Syll.
7.	Aa	3, Sep.
8.	Ma	6, 7, Mod. Pon.
9.	a=b	C.A. (denial of conclusion)
10.	Mb	8, 9, ID
11.	−Mb	3, Sep.
12.	Mb & −Mb	12, 13, Conj.
13.	a=b → (Mb & −Mb)	10→12, C.P.
14.	a≠b	13, Reductio

(8.3) 7.

1.	(∀x)(Tx → x=a)	
2.	(∀x)(Ux → x=b)	
3.	(∃x)(Tx & Ux)	⊢ a=b
4.	Tc & Uc	3, E.I. choosing a new name *c*
5.	Tc	4, Sep.
6.	Uc	4, Sep.
7.	Tc → c=a	1, U.I. using the *c*
8.	c=a	7,5, Mod. Pon.
9.	Uc → c=b	2, U.I. using the *c*
10.	c=b	9, 6, Mod. Pon.
11.	a=b	8, 10, ID

(8.3) 11. Using *Px* : *x* is president of the company; *Tx*: *x* is an excellent tennis player; *a*: Annie

1.	$(\exists x)[Px \,\&\, (\forall y)(Py \rightarrow y=x) \,\&\, Tx$	
2.	$-Ta$	$\vdash\ -Pa$
3.	$Pb \,\&\, (\forall y)(Py \rightarrow y=b) \,\&\, Tb$	1, E.I. choosing a new name *b*
4.	Pb	3, Sep.
5.	$(\forall y)(Py \rightarrow y=b)$	3, Sep.
6.	Tb	3, Sep.
7.	$\quad\vert\ Pa$	C.A. (denial of conclusion)
8.	$\quad\vert\ Pa \rightarrow a=b$	5, U.I. using the *a*
9.	$\quad\vert\ a=b$	8, 7, Mod. Pon.
10.	$\quad\vert\ Ta$	6, 9, ID
11.	$\quad\vert\ Ta \,\&\, -Ta$	10, 2, Conj.
12.	$Pa \rightarrow (Ta \,\&\, -Ta)$	8→11, C.P.
13.	$-Pa$	12, Reductio

(8.3) 13. Using *Ax* : *x* is an art thief; *Hx*: *x* has a sense of humor; *c*: Crown; *t*: Thomas

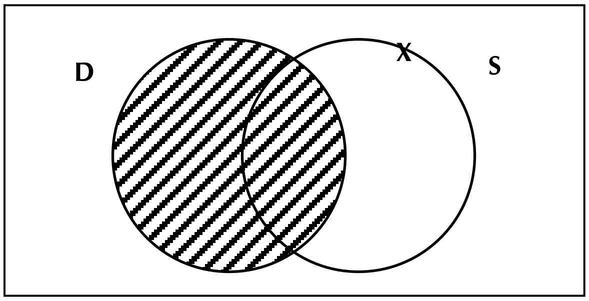
1.	$(\exists x)(Ax \,\&\, -Hx)$	
2.	$Ac \,\&\, (\forall x)(Ax \rightarrow x=c)$	
3.	$-Ht \,\&\, (\forall x)(-Hx \rightarrow x=t)$	$\vdash\ t=c$
4.	$Aa \,\&\, -Ha$	1, E.I. choosing a new name *a*
5.	$(\forall x)(Ax \rightarrow x=c)$	2, Sep.
6.	$Aa \rightarrow a=c$	5, U.I. using the *a*
7.	Aa	4, Sep.
8.	$a=c$	6, 7, Mod. Pon.
9.	$(\forall x)(-Hx \rightarrow x=t)$	3, Sep.
10.	$-Ha \rightarrow a=t$	9, U.I. using the *a*
11.	$-Ha$	4, Sep.
12.	$a=t$	10,11, Mod. Pon.
13.	$t=c$	12, 8, ID

Answers to Exercises in Appendix A

Exercise A.1

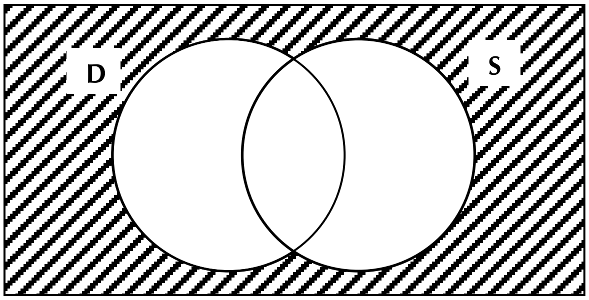
(A.1) 3. – (∃x)Dx & (∃x)(Sx ∨ – Sx)

The Universe



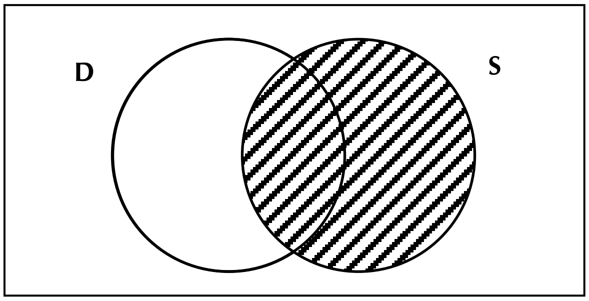
(A.1) 4. (∀x)(Dx ∨ Sx)

The Universe



(A.1) 5. – (∃x)Sx

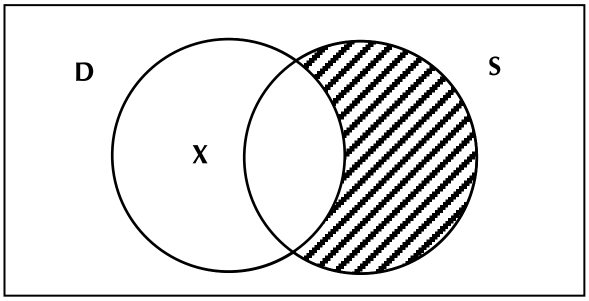
The Universe



Exercise A.2

(A.2) 2. Symbolically: (∃x)(Dx & – Sx), (∀x)(Sx → Dx) ⊢ (∃x)(Dx & Sx)

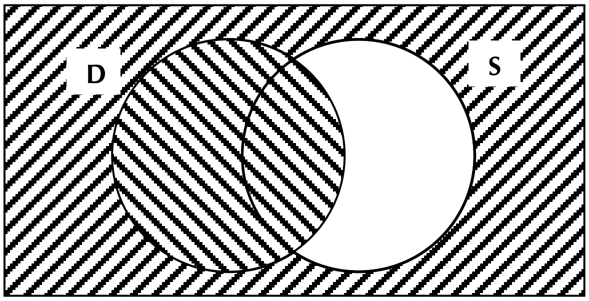
The Universe



Invalid

(A.2) 4. Symbolically: (∀x)(Dx ∨ Sx), – (∃x)Dx [or: (∀x) – Dx] ⊢ (∀x)Sx

The Universe

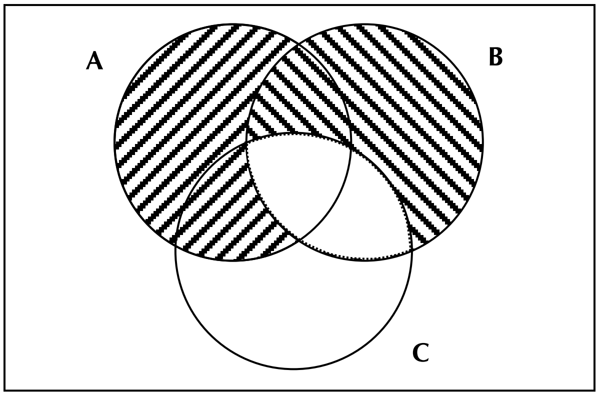


Valid

Exercise A.3

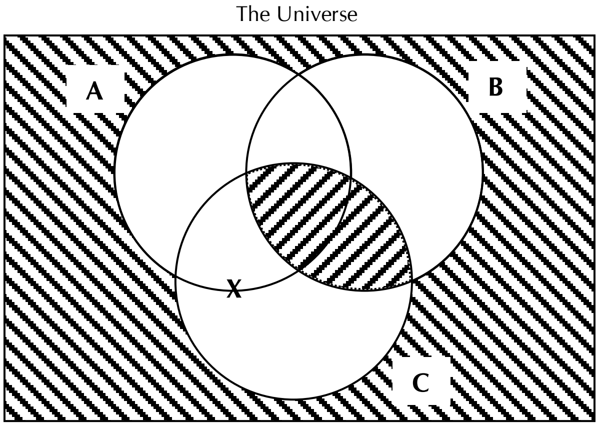
(A.3) 1. Symbolically: (∀x)(Ax → Bx), (∀x)(Bx → Cx) ⊢ (∀x)(Ax → Cx)

The Universe



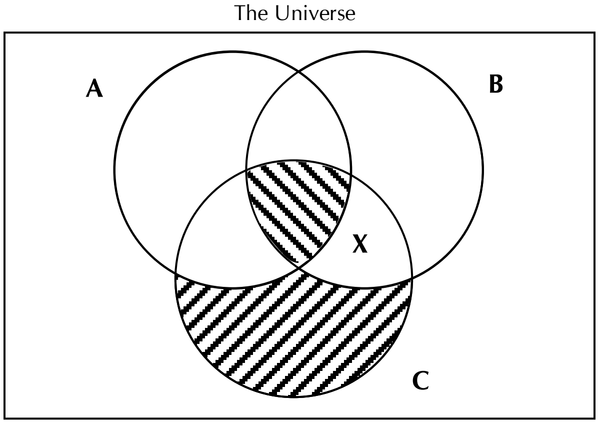
Valid

(A.3) 3. Symbolically: $(\forall x)(Ax \lor Bx \lor Cx)$, $(\forall x)(Bx \rightarrow -Cx)$, $(\exists x)Cx \vdash (\exists x)(Ax \& Cx)$



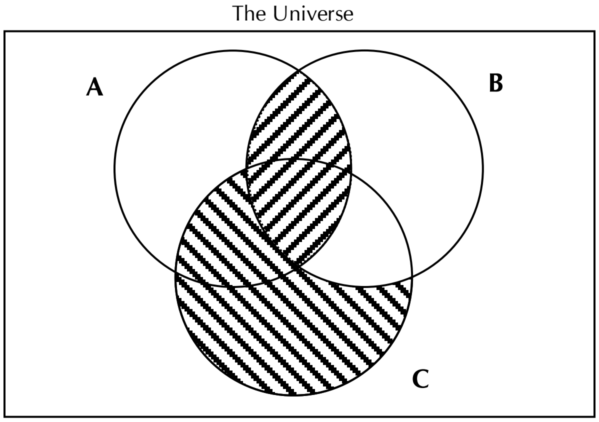
Invalid, because the "x" would have to be unambiguously in the *C-A* overlap area in order to represent the conclusion; but it isn't definitely there.

(A.3) 8. Symbolically: $(\forall x)[(Ax \& Bx) \rightarrow -Cx]$, $(\exists x)(Cx \& -Ax)$, $(\forall x)[Cx \rightarrow (Ax \lor Bx)] \vdash (\exists x)Bx$



Valid

(A.3) 10. Symbolically: $(\forall x)(-Bx \rightarrow -Cx)$ [or: $(\forall x)(Cx \rightarrow Bx)$], $(\forall x)(Ax \rightarrow -Bx) \vdash (\forall x)(Ax \rightarrow -Cx)$



Valid

Glossary

Symbols:

&	AND.
∨	OR (inclusive: “p or q or both”).
⊕	OR (exclusive: “p or q but not both”).
→	Conditional: IF ... THEN.
↔	Biconditional: IF AND ONLY IF.
−	NOT (denial).
⊢	Therefore.
|	ALTERNATIVE DENIAL (NAND).
↓	JOINT DENIAL (NOR).
≡	Logical equivalence.
=	Identity.
≠	Non-identity (difference).
∀	Universal quantifier.
∃	Existential quantifier.

Absorption: An elementary argument form.

Alternative Denial: A connective which creates a compound sentence which is true just in case one or the other (or both) of the constituent sentences is false. Also called the *NAND* function.

AND: A truth-functional connective which creates a *conjunction* out of two component sentences, called *conjuncts*, in such a way that the conjunction is true only when both conjuncts are true.

Antecedent: The “if” part of an “if...then” sentence. *See* **Conditional**.

Argument: A set of sentences, one of which, the conclusion, is said to follow from the others, the premises. If the conclusion cannot possibly be false when the premises are true, then the argument is said to be *deductively valid*; otherwise, the argument is either *deductively invalid*, or else it is an *inductive* argument which does not even pretend to establish the conclusion with certainty.

Argument Form: A generalized pattern of inference, which can be *instantiated* by different particular arguments.

Associative Laws: A Rule of Substitution (Equivalence).

Asymmetry: A property of some relations such that if the relation holds between one thing and a second, then it cannot hold between the second and the first.

Atomic Sentence: A single sentence letter or its denial; non-compound sentence.

Biconditional: A sentence of the form "*p* if and only if *q*", and which is true only when *p* and *q* have the same truth value. A biconditional is logically equivalent to the conjunction of two conditional sentences, namely, "if *p* then *q*" and "if *q* then *p*".

Branch: A section of a *Truth Tree* whose consistency conditions are independent of other branches in the tree.

Closed Branch: A branch of a truth tree in which there are two contradictory atomic sentences; a branch in which the sentences do not form a consistent set.

Commutative Laws: A Rule of Substitution (Equivalence).

Compound Sentence: A sentence composed of two or more sentences connected with some connective (operator); non-atomic sentence.

Conditional: A sentence of the form "If *p* then *q*". The *p* is called the *antecedent*, and the *q* is called the *consequent*. A conditional is false only in case its antecedent is true and its consequent is false. Also called *hypothetical*.

Conditional Proof: A procedure within the Method of Derivation wherein a premise is conditionally (hypothetically) introduced, in consequence of which a conditional sentence is derived.

Conjunction: An Elementary Argument Form. Also *see* **AND**.

Conjunctive Normal Form: A sentence expressed as a conjunction of disjunctions. Also known as *Product of Sums*.

Connective: A function which creates a compound sentence out of simpler sentences (atomic or compound). A connective is *truth functional* if it completely determines the truth value of a compound sentence based upon the truth values of the constituent sentences. Also called *operator*. (The dash, "−", is, strictly speaking, not a connective, since it does not connect sentences together.)

Consequent: The "then" part of an "if...then" sentence. *See* **Conditional**.

Consistent: A set of sentences is consistent if there is at least one way to assign truth values to the atomic elements of the sentences such that all the sentences in the set are true. In the case of a single sentence (i.e., a set with only one member), this means that a sentence is consistent if there is any way that it can be true. A consistent sentence is either contingent or tautologous.

Constant: *See* **Name**.

Contingent: A sentence is contingent if there is some way to make it true *and* some way to make it false; i.e., it is neither tautologous nor contradictory. A contingent sentence is also called *logically indeterminate*.

Contradiction: *See* **Inconsistent**.

Contradictories: Two sentences which must have opposite truth values.

Contraposition: A Rule of Substitution (Equivalence).

Contraries: Two sentences which might both be false, but which cannot both be true. *See* also **Subcontraries**.

Counterexample: A set of truth values for the atomic sentences in an argument such that the premises are true and the conclusion is false.

Countertautology: *See* **Inconsistent**.

Deduction: The inferring of a conclusion from premises with the intention that the truth of the conclusion be absolutely guaranteed by the truth of the premises.

DeMorgan's Laws: A Rule of Substitution (Equivalence).

Denial: *See* **NOT**.

Derivation: The *Method of Derivation* is a set of rules for deducing conclusions from given premises.

See **Deduction**.

Dilemma: An Elementary Argument Form.

Direct Derivation: The Method of Derivation without Indirect Proof.

Disjunctive Normal Form: A sentence expressed as a disjunction of conjunctions. Also called *Sum of Products.*

Disjunctive Syllogism: An Elementary Argument Form.

Distributive Laws: A Rule of Substitution (Equivalence).

Double Negation: A Rule of Substitution (Equivalence).

Dropping a Quantifier: *See* **Universal Instantiation** and **Existential Instantiation**.

Duality Rules: Rules for substituting universal quantifiers for existential quantifiers, or *vice-versa.*

Dyadic: Having two terms.

Elementary Argument Form: An argument form chosen for its simplicity and usefulness.

Elimination of Contradiction (E.C.): An (unofficial) inference rule by means of which a contradiction may be eliminated from a disjunction.

Elimination of Tautology (E.T.): An (unofficial) inference rule by means of which a tautology may be eliminated from a conjunction.

Equivalence: *See* **Logical Equivalence**.

Equivalence Rules: Rules which allow the substitution of one kind of sentence for another. Also called *Rules of Substitution* and *Rewriting Rules.*

Exclusive OR: *See* **OR**.

Existential Generalization (E.G.): A rule for quantifying (existentially) over an expression containing constants.

Existential Instantiation (E.I.): A rule for dropping existential quantifiers: *Rewrite the sentence without the quantifier, replacing each instance of its variable with some* ***new*** *name.*

Existential Quantifier: *See* **Quantifier**.

Exportation: A Rule of Substitution (Equivalence).

Fallacy: An invalid argument. Often the term is reserved for an argument which seems valid but actually is not.

Formal Proof of Validity: The derivation of a conclusion from given premises using only the Elementary Argument Forms and Equivalences.

Gate: An electronic circuit which performs a logic function (such as AND, OR, etc.).

Generalization: Quantifying a propositional function or an expression containing constants or quasi-variables. *See also* **Existential Generalization** and **Universal Generalization**.

General Rule: A rule for using the Tree Method with quantified expressions: *A quantifier may not be dropped unless the scope of the quantifier includes the* ***entire*** *sentence (i.e., not even preceded by a denial sign).*

Hypothetical: *See* **Conditional**.

Hypothetical Syllogism: An Elementary Argument Form.

Idempotency: A Rule of Substitution (Equivalence).

Iff: IF AND ONLY IF.

Inclusive OR: *See* **OR**.

Inconsistent: A set of sentences is inconsistent if there is no way to assign truth values to the atomic elements of the sentences such that all the sentences in the set are true. In the case of a single sentence (i.e., a set with only one member), it is inconsistent if there is no way that it can be true. An inconsistent sentence is also called *contradictory*, *countertautologous*, or *logically false*.

Identity: A relation between two names such that the relation is true if the two names name the same entity.

Indirect Proof: A proof procedure wherein the desired conclusion is denied, and in consequence a contradiction is produced. Also called *Proof by Contradiction.*

Induction: The inferring of a conclusion but without assuming that the truth of the premises absolutely guarantees the truth of the conclusion.

Infinite Branch: A branch of a tree which will continue to grow and never close. Infinite branches are treated as open branches.

Instance: An example or member of a general form.

Infix Notation: A method of writing expressions such that the operators are written between the operands. Precedence rules and/or parentheses are necessary in order to disambiguate some expressions. *See* also **Prefix Notation** and **Postfix Notation**.

Instantiate: To produce an instance; to substitute a particular for a general. In quantification, it is the substituting of *names* (*constants*) for variables in propositional functions.

Intransitivity: A property of some relations which is true if, when the relation holds between one thing and a second, and between the second and a third, then the relation cannot hold between the first and the third.

Invalid: *See* **Valid**.

Irreflexivity: A property of some relations such that the relation does not hold between a thing and itself.

Joint Denial: A connective which creates a compound sentence which is true just in case both of the constituent sentences are false. Also known as the *NOR* function.

Law of Excluded Middle: A sentence may not have a value other than *True* or *False*.

Law of Identity: In any given context, a sentence may not change its truth value.

Law of Non-Contradiction: A sentence may not have both the values *True* and *False*.

Logical Equivalence: Two sentences are logically equivalent if they have identical columns in a truth table; i.e., if they share the same truth values under all conditions.

Logically Determinate: Either necessarily true or necessarily false; i.e., either tautologous or contradictory; i.e., non-contingent.

Logically False: *See* **Inconsistent**.

Logically Indeterminate: Neither necessarily true nor necessarily false; i.e., contingent.

Logically True: *See* **Tautology**.

Many-Valued Logic: A system of logic which uses more than two truth values.

Material Equivalence: A Rule of Substitution (Equivalence).

Material Implication: A Rule of Substitution (Equivalence).

Metalanguage: A language used to talk about another language.

Metalogical: Having to do with statements about a system of logic, as opposed to sentences in the system of logic itself.

Method of Truth Trees: *See* **Truth Trees**.

Modus Ponens: An Elementary Argument Form.

Modus Tollens: An Elementary Argument Form.

Name: An individual constant which picks out some specific entity in the universe. (As opposed to a *variable*.)

NAND: *See* **Alternative Denial**.

Necessarily False: *See* **Inconsistent**.

Necessarily True: *See* **Tautology**.

Necessary Condition: The consequent of a conditional.

Non-Reflexivity: A property of some relations such that the relation might or might not hold between a thing and itself.

Non-Symmetry: A property of some relations such that if the relation holds between one thing and a second, then it might or might not hold between the second and the first.

Non-Transitivity: A property of some relations which is true if, when the relation holds between one thing and a second, and between the second and a third, then the relation might or might not hold between the first and the third.

NOR: *See* **Joint Denial**.

NOT: A truth functional operator which reverses the truth value of the sentence it operates on.

Object Language: An application language; a language talked about or described by a *metalanguage*.

Open Branch: A branch of a truth tree in which there are no two atomic sentences which contradict each other; a branch in which all sentences form a consistent set.

Operand: That which is operated on.

Operator: Connective.

OR: A truth functional connective which creates a *disjunction* out of two component sentences, called *disjuncts*, in such a way that the disjunction is true if either or both disjuncts are true (for the *inclusive OR*), or if exactly one of the disjuncts is true (for the *exclusive OR*).

Polish Notation: Prefix Notation.

Postfix Notation: A method of writing expressions which avoids the necessity for precedence rules or parentheses. In postfix notation, an operator is written after its operands. *See* also **Infix Notation** and **Prefix Notation**.

Precedence Rules: A hierarchy of operators such that, in the absence of overrides such as parentheses, a higher precedence operator will be taken to be operating on its operands before any lower precedence operators.

Predicate: A property ascribed to some subject.

Prefix Notation: A method of writing expressions which avoids the necessity for precedence rules or parentheses. In prefix notation, an operator is written before its operands. *See* also **Infix Notation** and **Postfix Notation**.

Prime Directive: A rule for using the Tree Method with quantified expressions: *Unquantified compound sentences (even if their components are quantified)* ***must*** *be decomposed in the usual manner, whereas quantified sentences cannot be decomposed until the quantifier is "dropped".*

Product of Sums: *See* **Conjunctive Normal Form**.

Proof by Contradiction: *See* **Indirect Proof**.

Proposition: Sentence.

Propositional Function: A predicate-variable combination which has no truth value because it is not a sentence, but which can be made into a sentence either by *instantiation* (replacing the variables by *constants*) or by *generalization* (putting the propositional function within the scope of a quantifier).

Quantifier: A term designating the number of subjects to which a predicate is said to apply. The *universal quantifier*, ∀, refers to an entire class, whereas the *existential quantifier*, ∃, refers to at least one member of a class.

Quantifier Duality Rules: *See* **Duality Rules**.

Quasi-Variable: A special notational convenience to represent an item otherwise unnamed but chosen at random and therefore being a representative of the universal class.

Reductio: Short for **Reduction to Absurdity**.

Reduction to Absurdity: An Elementary Argument Form.

Reflexivity: A property of some relations such that the relation holds between a thing and itself.

Relation: A two or more place predicate.

Repetition: An Elementary Argument Form.

Restrict the Universe of Discourse: Restrict the class of things which may instantiate variables.

Reverse Polish Notation: Postfix Notation.

Rewriting Rules: *See* **Equivalence Rules**.

Rules of Substitution: *See* **Equivalence Rules**.

Scope: The propositional functions (along with any operators which affect them) governed by a quantifier.

Sentence: A claim, statement, or proposition which must be either true or false.

Sentence Function: *See* **Propositional Function**.

Separation: An Elementary Argument Form.

Sound: A deductive argument is said to be sound if (1) it is valid and (2) its premises are, as a matter of fact, true.

Sheffer Stroke: *See* **Alternative Denial**.

Simplification of Contradiction (S.C.): An (unofficial) inference rule by means of which one of the conjuncts of a conjunction may be eliminated if the other conjunct is a contradiction.

Simplification of Tautology (S.T.): An (unofficial) inference rule by means of which one of the disjuncts of a disjunction may be eliminated if the other disjunct is a tautology.

Subcontraries: Two sentences which might both be true, but which cannot both be false. *See* also **Contraries**.

Subject: The entity of which some property is predicated.

Sufficient Condition: The antecedent of a conditional.

Sum of Products: *See* **Disjunctive Normal Form**.

Syllogism: An argument having two premises and a conclusion.

Symmetry: A property of some relations such that if the relation holds between one thing and a second, then it also holds between the second and the first.

Tautological Derivation: The Method of Derivation used to derive a tautology from any (or no) premises at all.

Tautology: A sentence which is true no matter what truth values its constituent sentences have. Also called *logically true* or *necessarily true*.

Transitivity: A property of some relations such that if the relation holds between a first and a second thing, and between the second and a third, then it holds between the first and the third.

Tree Method: *See* **Truth Trees**.

Triadic: Having three terms.

Truth Functional: A connective (operator) is truth functional if the truth value of a compound sentence employing that connective is a function strictly of the truth values of the component sentences.

Truth Table: A matrix which lays out all possible truth values for a given sentence, based upon the possible combinations of truth values for the constituent sentences and the definitions of any operators used.

Truth Trees: A diagrammatic tool used to display the consistency conditions of a sentence (or set of sentences) by decomposing compound sentences into simpler elements.

Truth Value: A value which a sentence may have. In binary (bivalent) logics, the values are *True* and *False*. In multi-valued logics, additional values may be available.

Universal Generalization (U.G.): A rule for quantifying, universally, over an expression containing quasi-variables (and containing no constants introduced by E.I. or C.A.).

Universal Instantiation (**U.I.**): A rule for dropping universal quantifiers: *Rewrite the sentence without the quantifier, replacing every occurrence of its variable by an individual constant (name) which occurs in the present context. If no names occur, make one up.*

Universal Quantifier: *See* **Quantifier**.

Universe of Discourse: The class of things which may instantiate variables.

Valid: A deductive argument is valid if and only if there is no possible way that the premises could be true and the conclusion false. Otherwise, the argument is *invalid*.

Variable: Part of a propositional function which is a place holder for *names* (*constants*).

Venn Diagram: A graphical method of representing quantified propositions.

Weakening: An Elementary Argument Form.

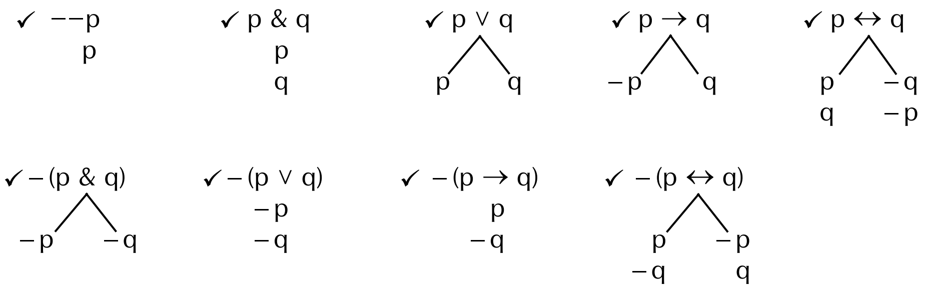
The Elementary Argument Forms

Absorption	$p \rightarrow (p \& q)$	$\vdash p \rightarrow q$
Conjunction	p, q	$\vdash p \& q$
Dilemma	$p \rightarrow q, r \rightarrow s, p \vee r$	$\vdash q \vee s$
	$(p \rightarrow q) \& (r \rightarrow s), p \vee r$	$\vdash q \vee s$
Disjunctive Syllogism	$p \vee q, -p$	$\vdash q$
	$p \vee q, -q$	$\vdash p$
Hypothetical Syllogism	$p \rightarrow q, q \rightarrow r$	$\vdash p \rightarrow r$
Modus Ponens	$p \rightarrow q, p$	$\vdash q$
Modus Tollens	$p \rightarrow q, -q$	$\vdash -p$
Reduction to Absurdity	$p \rightarrow (q \& -q)$	$\vdash -p$
	$p \rightarrow q, p \rightarrow -q$	$\vdash -p$
Repetition	p	$\vdash p$
Separation	$p \& q$	$\vdash p$
	$p \& q$	$\vdash q$
Weakening	p	$\vdash p \vee q$

The Equivalences (Rules of Substitution)

Associative Laws	$p \& (q \& r)$	$\equiv (p \& q) \& r$
	$p \vee (q \vee r)$	$\equiv (p \vee q) \vee r$
Commutative Laws	$p \& q$	$\equiv q \& p$
	$p \vee q$	$\equiv q \vee p$
Contraposition	$p \rightarrow q$	$\equiv -q \rightarrow -p$
DeMorgan's Laws	$-(p \& q)$	$\equiv -p \vee -q$
	$-(p \vee q)$	$\equiv -p \& -q$
Distributive Laws	$p \& (q \vee r)$	$\equiv (p \& q) \vee (p \& r)$
	$p \vee (q \& r)$	$\equiv (p \vee q) \& (p \vee r)$
Double Negation	p	$\equiv - - p$
Exportation	$(p \& q) \rightarrow r$	$\equiv p \rightarrow (q \rightarrow r)$
Idempotency	p	$\equiv p \& p$
	p	$\equiv p \vee p$
Material Equivalence	$p \leftrightarrow q$	$\equiv (p \rightarrow q) \& (q \rightarrow p)$
	$p \leftrightarrow q$	$\equiv (p \& q) \vee (-p \& -q)$
Material Implication	$p \rightarrow q$	$\equiv -p \vee q$

Decomposition Rules for Truth Trees



Duality Rules

$(\forall x) \equiv -(\exists x)-$　　　$-(\forall x) \equiv (\exists x)-$

$(\exists x) \equiv -(\forall x)-$　　　$-(\exists x) \equiv (\forall x)-$

Universal Instantiation (U.I.)

$(\forall x)\Phi x$

Φa

where Φx is any propositional function using the variable x (or y, etc.), and a (or b, etc.) is any name (constant) which already appears in the present context, substituted for all instances of x. (If no name exists, invent one.) In the method of derivation, an instantiation may also take the form of a quasi-variable (“1”, “2”, etc.) if a subsequent Universal Generalization is anticipated.

Existential Instantiation (E.I.)

✓ $(\exists x)\Phi x$

$\Phi \mathrm{a}$

where Φx is any propositional function using the variable x (or y, etc.), and a (or b, etc.) is any *new* constant (i.e., a name which does not yet appear in the present context) which is substituted for all instances of x.

Universal Generalization (U.G.)

$\Phi\#$

$(\forall x)\Phi x$

where $\Phi\#$ is any proposition containing quasi-variable # (“1”, “2”, etc.) and not containing a constant introduced via E.I. or C.A., and Φx is any propositional function where the variable x (or y, etc.) has replaced all instances of #.

Existential Generalization (E.G.)

Φa

$(\exists x)\Phi x$

where Φa is any proposition using the constant a (or b, etc.), and Φx is any propositional function using the variable x (or y, etc.) to replace all instances of a.